Homework #8 Solutions

ECE 603 Fall, 2016

1) (a) $P_x = R_x(0) = 1$

(b) Yes. WSS, Gaussian $\Rightarrow$ SSS

(c) Let $R = X(3)$. Because $X(t)$ is a Gaussian RP, R is Gaussian.

$$E[R] = E[X(3)] = 0$$

$$\sigma_R^2 = E[R^2] = E[X^2(3)] = E[X^2(t)] = R_x(0) = 1$$

$$P(X(3) > 2) = P(R > 2) = 1 - P(R \le 2)$$

$$= 1 - \left(\tfrac{1}{2} + \tfrac{1}{2}\operatorname{erf}(\sqrt{2})\right) = \tfrac{1}{2} - \tfrac{1}{2}\operatorname{erf}(\sqrt{2})$$

(d)

$$S_x(f) = \mathcal{F}\{R_x(\tau)\}$$

$$= \mathcal{F}\{e^{-|\tau|/4}\}$$

$$= \int_{-\infty}^{\infty} e^{-|\tau|/4} e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{-\infty}^{0} e^{\tau/4} e^{-j2\pi f\tau}\, d\tau + \int_{0}^{\infty} e^{-\tau/4} e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{-\infty}^{0} e^{+(1/4 - j2\pi f)\tau}\, d\tau + \int_{0}^{\infty} e^{-(1/4 + j2\pi f)\tau}\, d\tau$$

$$= \frac{1}{1/4 - j2\pi f} e^{(1/4 - j2\pi f)\tau}\Big|_{-\infty}^{0} + \frac{-1}{1/4 + j2\pi f} e^{-(1/4 + j2\pi f)\tau}\Big|_{0}^{\infty}$$

$$= \frac{1}{1/4 - j2\pi f} + \frac{1}{1/4 + j2\pi f}$$

$$= \frac{1/4 + j2\pi f + 1/4 - j2\pi f}{1/16 + 4\pi^2 f^2}$$

$$= \frac{8}{1 + 64\pi^2 f^2}$$

(e) $S_X(f) = \frac{8}{1+64\pi^2 f^2} = \frac{2\sqrt{2}}{1-j8\pi f} \frac{2\sqrt{2}}{1+j8\pi f}$

$$\Rightarrow H(f) = \frac{2\sqrt{2}}{1-j8\pi f}\sqrt{\frac{2}{N_0}} = \frac{4}{\sqrt{N_0}(1-j8\pi f)}$$

(f) $Z = X(0) + X(1) + X(2)$

Because $X(t)$ is a Gaussian RP, $X(0)$, $X(1)$, and $X(2)$ are jointly Gaussian $\Rightarrow$ Z is a linear combination of jointly Gaussian RVs $\Rightarrow$ Z is Gaussian

$$E[Z] = E[X(0)] + E[X(1)] + E[X(2)] = 0$$

$$\sigma_Z^2 = E[Z^2] - (E[Z])^2 = E[Z^2] = E[(X(0) + X(1) + X(2))^2]$$

$$= E[X^2(0)] + E[X^2(1)] + E[X^2(2)] + 2E[X(0)X(1)]$$
$$+ 2E[X(1)X(2)] + 2E[X(0)X(2)]$$
$$= 3R_X(0) + 4R_X(1) + 2R_X(2)$$
$$= 3 + 4e^{-1/4} + 2e^{-1/2}$$

$$\Longrightarrow \quad f_z(z) = \frac{1}{\sqrt{2\pi(3+4e^{-1/4}+2e^{-1/2})}}\, e^{-z^2/2(3+4e^{-1/4}+2e^{-1/2})}$$

(g)

Let $T = X(0) + X(1)$. As in (f), T is Gaussian with $E[T] = 0$ and

$$\sigma_T^2 = E[T^2] - (E[T])^2 = E[T^2]$$
$$= E[X^2(0)] + E[X^2(1)] + 2E[X(0)X(1)]$$
$$= 2R_x(0) + 2R_x(1) = 2 + 2e^{-1/4}$$

$$P(T > 3) = 1 - P(T \le 3)$$
$$= 1 - \left(\tfrac{1}{2} + \tfrac{1}{2}\operatorname{erf}\left(\frac{3}{\sqrt{2+2e^{-1/4}}}\right)\right)$$
$$= 1/2 - \tfrac{1}{2}\operatorname{erf}\left(\frac{3}{\sqrt{2+2e^{-1/4}}}\right)$$

IGNORE

(h) $\quad Y = \int_0^T N(t)\,dt$

$$N(t) \longrightarrow \boxed{h(\cdot)} \longrightarrow m(t)$$

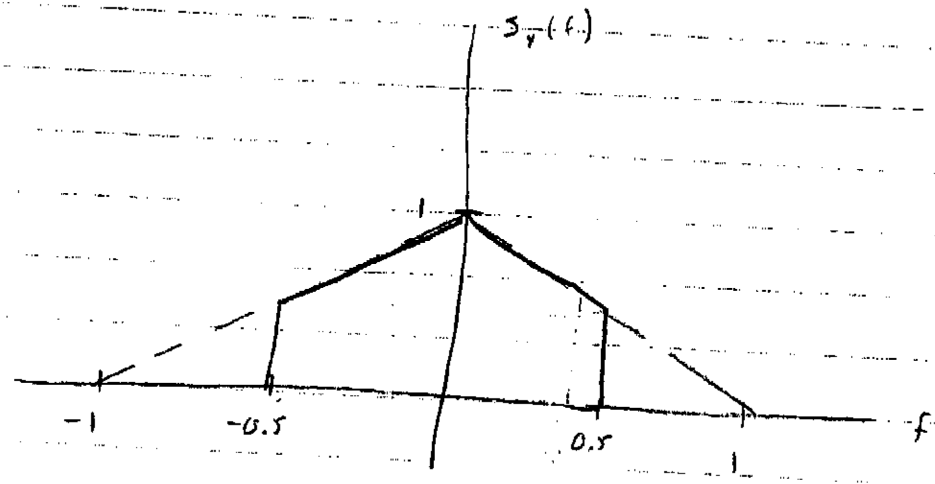
$$h(\tau) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{else} \end{cases}$$

$m(t)$ is a Gaussian RP (since $N(t)$ is a Gaussian RP).

$\Longrightarrow Y = m(T)$ is Gaussian

$$E[Y] = E\left[\int_0^T N(t)\,dt\right] = \int_0^T E[N(t)]\,dt = 0$$

2) (a) $S_Y(f) = |H(f)|^2 S_X(f)$



$$\Rightarrow S_Y(f) = \begin{cases} 1 - |f| , & |f| \le 0.5 \\ 0 , & \text{else} \end{cases}$$

$$P_Y = \int_{-\infty}^{\infty} S_Y(f)\,df = 0.5 + 2\left(\tfrac{1}{2} \cdot 0.5 \cdot 0.5\right) = 0.75$$

Because $X(t)$ is Gaussian, so is $Y(t)$

$\Rightarrow Y(0)$ is Gaussian

$E[Y(0)] = 0 \qquad Var[Y(0)] = E[Y^2(0)] - (E[Y(0)])^2$

$$= E[Y^2(0)]$$

$$= R_Y(0) = P_Y = 0.75$$

$$P(Y(0) \ge 2) = 1 - P(Y < 2)$$

$$= 1 - \left(\frac{1}{2} + \frac{1}{2} erf\left(\frac{2}{\sqrt{2}\sqrt{3/4}}\right)\right)$$

$$= \frac{1}{2} - \frac{1}{2} erf\left(2\sqrt{2}/\sqrt{3}\right)$$

(b) $H(f) = j2\pi f$

$$\Rightarrow H(f)H^*(f) = j2\pi f(-j2\pi f) = 4\pi^2 f^2$$

$$\Rightarrow S_Y(f) = S_X(f)\,|H(f)|^2 = \begin{cases} (1-|f|)\,4\pi^2 f^2 & |f| \le 1 \\ 0 & \text{else} \end{cases}$$

$$P_Y = \int_{-\infty}^{\infty} S_Y(f)\,df$$

$$= 2\int_0^1 (1-f)\,4\pi^2 f^2\,df$$

$$= 8\pi^2 \int_0^1 (f^2 - f^3)\,df$$

$$= 8\pi^2\left(1/3 - 1/4\right)$$

$$= 2/3\,\pi^2$$

$$P[Y(0) \ge 2] = 1 - P(Y(0) < 2)$$

$$= 1 - \left(\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}}{\sqrt{2/3\,\pi^2}}\right)\right)$$

$$= 1/2 - \frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{2}}{\sqrt{2/3\,\pi^2}}\right)$$

$$= 1/2 - \frac{1}{2}\operatorname{erf}\left(\frac{1}{\sqrt{\pi^2/3}}\right)$$

3) $S_X(f) = 10$

(a) $S_Y(f) = \begin{cases} 10, & |f| < 3 \\ 0, & \text{else} \end{cases}$

$\Rightarrow P_Y = \int_{-\infty}^{\infty} S_Y(f)\,df = 60$

(b)
$S_Z(f) = \begin{cases} 10e^{-4f^2}, & |f| < 3 \\ 0, & \text{else} \end{cases}$

$$P_Z = \int_{-3}^{3} 10\, e^{-4f^2}\, df$$

$$= \int_{-3}^{3} 10\, e^{-f^2/2\cdot(1/8)}\, df$$

$$= 10\cdot\sqrt{2\pi(1/8)} \int_{-3}^{3} \frac{1}{\sqrt{2\pi(1/8)}}\, e^{-f^2/2\cdot(1/8)}\, df$$

$$= 5\sqrt{\pi}\,\big(P(Z \le 3) - P(Z \le -3)\big) \quad \text{where } Z \sim N(0, 1/8)$$

$$= 5\sqrt{\pi}\,\left(\cancel{1/2} + 1/2\,\mathrm{erf}(6) - \left(\cancel{1/2} - 1/2\,\mathrm{erf}(6)\right)\right) \quad \frac{x-\mu}{\sqrt{2}\sigma} = \frac{3}{\sqrt{2}\cdot 1/2\sqrt{2}} = 6$$

$$= 5\sqrt{\pi}\,\mathrm{erf}(6)$$

4) (a)

$$P_x = R_x(0) = 20$$

(b)

$$m_Y(t) = E\left[4\cos(2\pi 30 t) + X(t)\right]$$

$$= 4\cos(2\pi 30 t) + \cancel{E[X(t)]}^{\to 0}$$

$$= 4\cos(2\pi 30 t)$$

$$R_Y(t_1, t_2) = E\left[Y(t_1)Y(t_2)\right]$$

$$= E\Big[\big(4\cos(2\pi 30 t_1) + X(t_1)\big)$$

$$\big(4\cos(2\pi 30 t_2) + X(t_2)\big)\Big]$$

$$= 16\cos(2\pi 30 t_1)\cos(2\pi 30 t_2)$$

$$+ 4\cos(2\pi 30 t_1)\,\cancel{E[X(t_1)]}^{\to 0}$$

$$+ 4\cos(2\pi 30 t_2)\,\cancel{E[X(t_2)]}^{\to 0}$$

$$+ E[X(t_1)X(t_2)]$$

$$= 16\cos(2\pi 30 t_1)\cos(2\pi 30 t_2) + 20\,\mathrm{sinc}^2(80(t_2 - t_1))$$

$m_Y(t)$ <u>not</u> constant, <u>not</u> WSS

(c)

<u>Yes</u>

Consider any $Y(t_1), Y(t_2), \ldots, Y(t_N)$

$$Z = \sum_{i=1}^{N} a_i Y(t_i) = \underbrace{4\sum_{i=1}^{N} a_i \cos(2\pi t_i)}_{\text{constant}} + \underbrace{\sum_{i=1}^{N} a_i X(t_i)}_{\text{Gaussian - since } X(t_1), X(t_2), \ldots, X(t_N) \text{ jointly Gaussian}}$$

$\Rightarrow$ Z is Gaussian

$\Rightarrow$ by Cramer-Wald Device, $Y(t_1), Y(t_2), \ldots, Y(t_N)$ are jointly Gaussian

$\Rightarrow Y(t)$ is a Gaussian RP

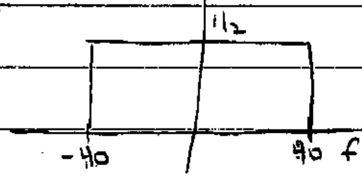
(d)

$$Z(t) = h(t) * Y(t)$$

$$= h(t) * \left(4\cos(2\pi 30 t) + X(t)\right)$$

$$= h(t) * 4\cos(2\pi 30 t) + h(t) * X(t)$$

Now, $H(f) = 1/2\, p(f/80)$



$$\Rightarrow \quad z(t) = 2\cos(2\pi 30 t) + u(t) \qquad u(t) \triangleq h(t) * X(t)$$

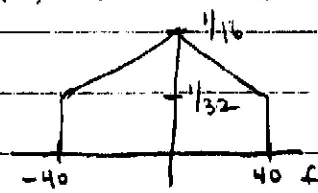
$$P_Z = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E\left[\left(2\cos(2\pi 30 t) + u(t)\right)^2\right] dt$$

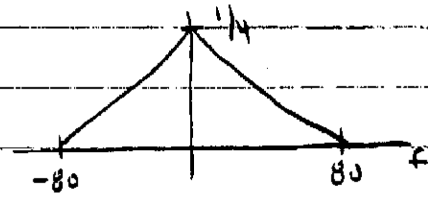
$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \Big(4\cos^2(2\pi 30 t) + 4\cos(2\pi 30 t)\, \cancelto{0}{E[u(t)]}$$
$$+ E[u^2(t)]\Big)\, dt$$

$$= \lim_{T\to\infty} \frac{4}{2T}\int_{-T}^{T} \left(1/2 + 1/2\cos(2\pi 60 t)\right) dt \quad + E[u^2(t)]$$

$$= 2 + E[u^2(t)]$$

Now, $S_u(f) = |H(f)|^2 S_X(f)$ where $S_X(f) = 1/4\, \Lambda(f/80)$





$$P_u = \int_{-\infty}^{\infty} S_u(f)\, df = 80 \cdot 1/32 + 80 \cdot 1/32 \cdot 1/2$$
$$= 5/2 + 5/4 = 15/4$$

$$\Rightarrow P_Z = 23/4$$