Homework #7 Solutions

ECE 603 — Fall, 2016

1) (a)

X_i: outcome of i^{th} toss

$$P(X_i = j) = \begin{cases} 1/6, & j = 1,2,3,4,5,6 \\ 0, & \text{else} \end{cases}$$

$$E[X_i] = \sum_{j=1}^{6} j \cdot 1/6 = 7/2$$

$$E[X_i^2] = \sum_{j=1}^{6} j^2 \cdot 1/6 = 91/6 \Rightarrow Var(X_i) = 91/6 - (7/2)^2 = 35/12$$

• Let $Y = \sum_{i=1}^{100} X_i$ $\quad E[Y] = 100(7/2) = 350$

$$Var[Y] = 100(35/12) = 1750/6$$

$$P(316 \le Y \le 384) = P(|Y-350| \le 34)$$
$$= 1 - P(|Y-350| \ge 34)$$
$$\ge 1 - \frac{1750/6}{(34)^2} = 0.747$$

• $$P\left(316 \le \sum_{i=1}^{100} X_i \le 384\right) = P\left(\sum_{i=1}^{100} X_i \le 384\right) - P\left(\sum_{i=1}^{100} X_i \le 316\right)$$
$$= P\left(\sum_{i=1}^{100} X_i - 350 \le 34\right) - P\left(\sum_{i=1}^{100} X_i - 350 \le -34\right)$$
$$= P\left(\frac{\sum_{i=1}^{100} X_i - 350}{\sqrt{100(35/12)}} \le 1.99\right) - P\left(\frac{\sum_{i=1}^{100} X_i - 350}{\sqrt{100(35/12)}} \le -1.99\right)$$
$$\approx \frac{1}{2} + \frac{1}{2}\,\text{erf}\left(1.99/\sqrt{2}\right) - \left(1/2 - \frac{1}{2}\,\text{erf}\left(1.99/\sqrt{2}\right)\right)$$
$$= \text{erf}\left(1.99/\sqrt{2}\right)$$
$$= 0.953$$

(b) $X_i = \begin{cases} 0, & \text{no error} \\ 1, & \text{error} \end{cases}$

$Y = \sum_{i=1}^{1000} X_i \qquad E[X_i] = 1 \cdot 0.15 + 0 \cdot 0.85 = 0.15$

$E[X_i^2] = 1^2 \cdot 0.15 + 0^2 \cdot 0.85 = 0.15 \Rightarrow Var(X_i) = 0.1275$

$$P(Y > 180) = P\left(\sum_{i=1}^{1000} X_i > 180\right)$$

$$= 1 - P\left(\sum_{i=1}^{1000} X_i \le 180\right)$$

$$= 1 - P\left(\sum_{i=1}^{1000} X_i - 150 \le 30\right)$$

$$= 1 - P\left(\frac{\sum_{i=1}^{1000} X_i - 150}{\sqrt{1000 \cdot 0.1275}} \le \underbrace{\frac{30}{\sqrt{1000 \cdot 0.1275}}}_{2.65}\right)$$

$$\approx 1 - \left(\frac{1}{2} + \frac{1}{2} \operatorname{erf}(2.65/\sqrt{2})\right)$$

$$= \frac{1}{2} - \frac{1}{2} \operatorname{erf}(2.65/\sqrt{2})$$

$$= 0.004$$

(c) Let X_i = number of message during second i.

Want $P(\sum_{i=1}^{60} X_i > 650)$.

$$\mu = E[X_i] = 10$$

$$\sigma^2 = E[X_i] = 10$$

$$P[\sum_{i=1}^{60} X_i > 650] = P[\sum_{i=1}^{60} X_i - 600 > 50]$$

$$= 1 - P\left(\frac{\sum_{i=1}^{60} X_i - 600}{\sqrt{60(10)}} \leq 50/\sqrt{600}\right)$$

$$\approx 1 - \left(\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(50/\sqrt{1200}\right)\right)$$

$$= \frac{1}{2} - \frac{1}{2}\operatorname{erf}(1.44)$$

$$= 0.02$$

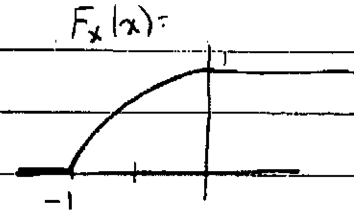
2)

(a) $P(X_0 \geq -1/2) = 1 - P(X_0 < -1/2)$

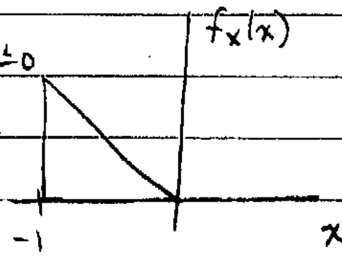
$= 1 - F_{X_0}(-1/2)$

$= 1 - (1 - (-1/2)^2)$

$= 1/4$



(b) $f_X(x) = d/dx\, F_X(x) = \begin{cases} -2x, & -1 \leq x \leq 0 \\ 0, & \text{else} \end{cases}$



$E[X] = \int_{-1}^{0} x(-2x)\,dx$

$= -2/3\, x^3 \Big|_{-1}^{0}$

$= -2/3$

(c) $E[X^2] = \int_{-1}^{0} x^2(-2x)\,dx = -2/4\, x^4 \Big|_{-1}^{0} = 1/2$

$Var(X) = 1/2 - (2/3)^2 = 1/18$

(d) $P(U \geq -115) = P\left(U - ((-2/3) \cdot 180) \geq -115 - (-2/3 \cdot 180)\right)$

$= P(U - 180\mu \geq 5)$

$= P\left(\frac{U - 120\mu}{\sqrt{180 \cdot 1/18}} \geq 5/\sqrt{10}\right)$

$= P(Z \geq 5/\sqrt{10})$

$= 1 - P(Z \leq 5/\sqrt{10})$

$\approx 1 - (1/2 + 1/2\, erf(5/\sqrt{20})) = 1/2 - 1/2\, erf(5/\sqrt{20})$

(e)

$$m_Y[n] = E[Y_n]$$
$$= E[X_n - X_{n-1}]$$
$$= E[X_n] - E[X_{n-1}]$$
$$= -2/3 - (-2/3)$$
$$= 0$$

(f)

$$R_Y[m,n] = E[Y_m Y_n]$$
$$= E[(X_m - X_{m-1})(X_n - X_{n-1})]$$
$$= E[X_m X_n] - E[X_{m-1} X_n] - E[X_{n-1} X_m] + E[X_{n-1} X_{m-1}]$$

If $|m-n|>1$,

$$R_Y[m,n] = (-2/3)^2 - (-2/3)^2 - (-2/3)^2 + (-2/3)^2 = 0$$

If $m=n$

$$R_Y[m,n] = 1/2 - (-2/3)^2 - (-2/3)^2 + 1/2$$
$$= 1/9$$

If $|m-n|=1$

$$R_Y[m,n] = (-2/3)^2 - (1/2)^2 - (2/3)^2 + (2/3)^2$$
$$= -1/18$$

$$R_Y[m,n] = \begin{cases} 1/9, & m=n \\ -1/18, & |m-n|=1 \\ 0, & \text{else} \end{cases}$$

yes. WSS!

3)

(a) X, Y independent, Gaussian

$\Rightarrow X, Y$ jointly Gaussian

$\Rightarrow Xt + Y$ Gaussian

$$E[Z(t)] = E[Xt] + E[Y] = 1$$

$$Var[Z(t)] \underset{\text{indep.}}{=} Var[Xt] + Var[Y] = 2t^2 + 1$$

$$\Rightarrow f_{Z(t)}(x) = \frac{1}{\sqrt{2\pi(2t^2+1)}} e^{-(x-1)^2/2(2t^2+1)}$$

(b)

$$m_X(t) = E[Z(t)] = 1 \qquad \text{(see above)}$$

(c)

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= E[(Xt_1 + Y)(Xt_2 + Y)]$$

$$= t_1 t_2 E[X^2] + t_1 \cancel{E[XY]}^{0} + t_2 \cancel{E[XY]}^{0} + E[Y^2]$$

$$= 2t_1 t_2 + 2$$

(d) No. $R_X(t_1, t_2)$ does not depend only on $|t_2 - t_1|$

4)

$$X(t) = A\cos(2\pi f_c t) + B\sin(2\pi f_c t)$$

(a)

$$E[X(t)] = E[A\cos(2\pi f_c t) + B\sin(2\pi f_c t)]$$

$$= \cancel{E[A]}^{\,0}\cos(2\pi f_c t) + \cancel{E[B]}^{\,0}\sin(2\pi f_c t)$$

$$= 0$$

(b)

$$E[X(t_1)X(t_2)]$$

$$= E[(A\cos(2\pi f_c t_1) + B\sin(2\pi f_c t_1))$$
$$(A\cos(2\pi f_c t_2) + B\sin(2\pi f_c t_2)]$$

$$= \cancel{E[A^2]}^{\,\sigma^2}\cos(2\pi f_c t_1)\cos(2\pi f_c t_2) + \cancel{E[AB]}^{\,0}\sin(2\pi f_c t_1)\cos(2\pi f_c t_2)$$
$$+ \cancel{E[AB]}^{\,0}\cos(2\pi f_c t_1)\sin(2\pi f_c t_2) + \cancel{E[B^2]}^{\,\sigma^2}\sin(2\pi f_c t_1)\sin(2\pi f_c t_2)$$

$$= \sigma^2(\cos(2\pi f_c t_1)\cos(2\pi f_c t_2) + \sin(2\pi f_c t_1)\sin(2\pi f_c t_2))$$

$$= \sigma^2\cos(2\pi f_c t_1 - 2\pi f_c t_2)$$

$$= \sigma^2\cos(2\pi f_c(t_1 - t_2))$$

(c) $X(t)$ is wide-sense stationary. $E[X(t)]$ is constant and $R_X(t_1, t_2)$ only depends on $t_1 - t_2$.

(d) $X(t)$ is <u>not</u> strict-sense stationary.

Consider

$$E[X^3(t)] = E[(A\cos(2\pi f_c t) + B\sin(2\pi f_c t))^3]$$

$$= E[A^3\cos^3(2\pi f_c t) + 3A^2B\cos^2(2\pi f_c t)\sin(2\pi f_c t) + 3B^2A\cos(2\pi f_c t)\sin^2(2\pi f_c t) + B^3\sin^3(2\pi f_c t)]$$

$$= E[A^3]\cos^3(2\pi f_c t) + 3E[A^2]\cancelto{0}{E[B]}\cos^2(2\pi f_c t)\sin(2\pi f_c t) + 3E[B^2]\cancelto{0}{E[A]}\cos(2\pi f_c t)\sin^2(2\pi f_c t) + E[B^3]\sin^3(2\pi f_c t)$$

$$\cancel{= E[A^3]\cos^3(2\pi f_c t) + E[B^3]\sin^3(2\pi f_c t)}$$

~~Assuming $E[A^3] \neq 0$ or $E[B^3] \neq 0$,~~
This depends on t, implying $f_{X(t)}(x)$ depends on t, implying

$$f_{X(t+\tau)}(x) \neq f_{X(t)}(x)$$

for some τ $\Rightarrow$ not strict-sense stationary.

5)

(a) $T \sim N(1, 4) \Rightarrow -T \sim N(-1, 4)$

$\Rightarrow \quad t - T \sim N(t-1, 4)$

(b)

$$E[X(t)] = E[t - T]$$
$$= E[t] - E[T]$$
$$= t - 1$$

$$E[X(t_1)X(t_2)] = E[(t_1 - T)(t_2 - T)]$$
$$= E[t_1 t_2] - E[t_1 T] - E[t_2 T] + E[T^2]$$
$$= t_1 t_2 - t_1 - t_2 + 5$$

(c) Consider any N, $(t_1, t_2, \ldots, t_N)$

$$Z = \sum_{i=1}^{N} \alpha_i X(t_i) = \sum_{i=1}^{N} \alpha_i (t_i - T) = \sum_{i=1}^{N} \alpha_i t_i - \left(\sum_{i=1}^{N} \alpha_i\right) T$$
$$\sim N\left(\sum_{i=1}^{N} \alpha_i t_i - \sum_{i=1}^{N} \alpha_i ,\ 4 \left(\sum_{i=1}^{N} \alpha_i \right)^2 \right)$$

$\Rightarrow X(t_1), X(t_2), \ldots, X(t_N)$ are jointly Gaussian $\Rightarrow X(t)$ is a Gaussian RP