Homework #4 Solutions

1) (a)

- $P(X<1) = F_X(1) - P(X=1)$
  $= 2/3 - 1/3 = 1/3$

- $P(1 \le X \le 1.5) = F_X(1.5) - P(X<1) = 5/6 - 1/3 = 1/2$

- $P(X=1) = \underbrace{F_X(1) - F_X(1^-)}_{\text{size of "jump"}} = 2/3 - 1/3 = 1/3$

(b) $f_X(x) = d/dx\, F_X(x)$

$$= \begin{cases} 0, & x<0 \\ 1/3, & 0 \le x < 2 \\ 0, & x \ge 2 \end{cases} + 1/3\,\delta(x-1)$$

(c) $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$= \int_0^2 x/3\,dx + \int_0^2 1 \cdot 1/3\,\delta(x-1)\,dx$$

$$= x^2/6 \Big|_0^2 + 1/3$$

$$= 2/3 + 1/3 = \underline{1}$$

$Var(X) = E[X^2] - (E[X])^2$

$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$

$$= \int_0^2 x^2/3\,dx + \int_0^2 1^2 \cdot 1/3\,\delta(x-1)\,dx$$

$$= x^3/9 \Big|_0^2 + 1/3$$

$$= 8/9 + 1/3$$

$$= 11/9$$

2) (a) $\int_{-2}^{1} cx^2\,dx = \frac{1}{3}cx^3 \Big|_{-2}^{1} = \frac{c}{3}(1+8) = 3c \Rightarrow c = \frac{1}{3}$

(b) $P(X^2 \ge 1) = P(X \le -1 \text{ or } X \ge 1)$

$= \int_{-2}^{-1} x^2/3 = x^3/9 \Big|_{-2}^{-1} = -1/9 + 8/9$

$= 7/9$

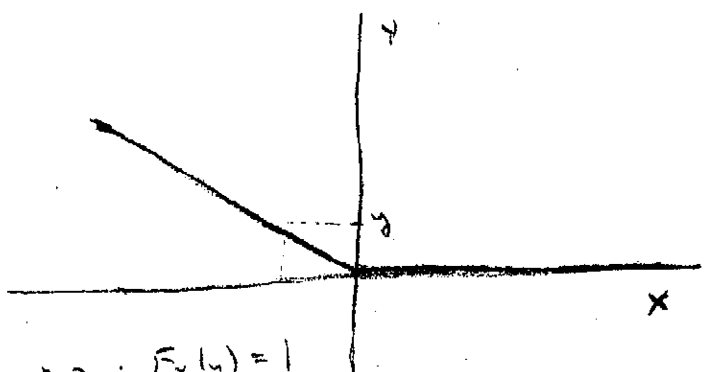
(c) $P(X-1 \ge -1/4) = P(X \ge 3/4)$

$= \int_{3/4}^{1} x^2/3\,dx$

$= x^3/9 \Big|_{3/4}^{1} = 1/9 - (3/4)^3/9$

$= \frac{1-(3/4)^2}{9}$

(d)



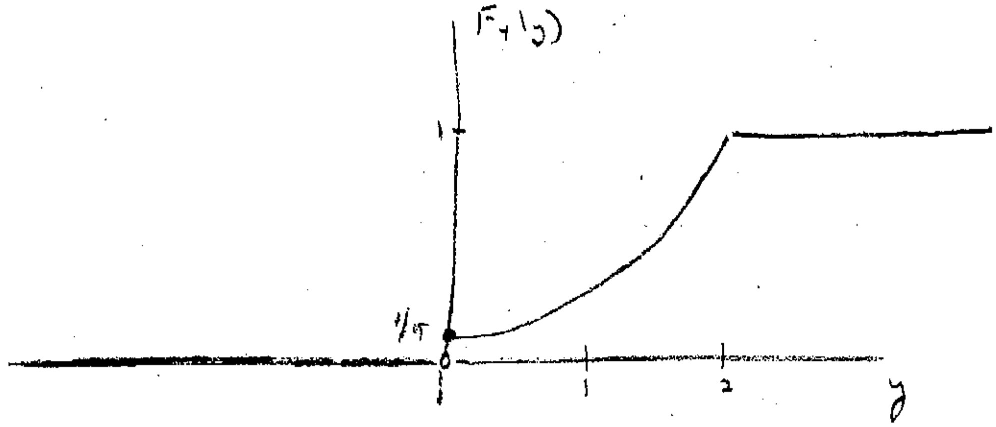
For $y \ge 2$: $F_Y(y) = 1$

For $y < 0$: $F_Y(y) = 0$

For $y = 0$: $P(Y=0) = P(X \ge 0) = \int_0^1 x^2/3 = 1/9$

For $2 \ge y \ge 0$: $F_Y(y) = P(Y \le y) = P(-X \le y) = P(X \ge -y) = 1/9 + \int_{-y}^{0} x^2/3\,dx$

$= 1/9 + x^3/9 \Big|_{-2}^{0} = 1/9 + y^3/9$



$$f_Y(y) = \tfrac{1}{9}\,\delta(y) + \begin{cases} 0, & y < 0, \\ y^2/3, & 0 < y < 2 \\ 0, & y > 2 \end{cases}$$

3)

(a) We know $P(0,1) = 1$

$$\tfrac{1}{2} + c\cdot\left((1-0) - \tfrac{1}{2}(1-0)^2\right)$$

$$= \tfrac{1}{2} + \tfrac{1}{2}c$$

$$\Rightarrow c = 1$$

~~(b) on next page~~

(c) $S_1 = \{A, B, C, D\}$

$$\mathcal{Y} = \mathcal{P}_{S_4} = \{\phi, \{A\}, \{B\}, \{C\}, \{D\}, \{A,B\}, \{A,C\}, \{A,D\}, \{B,C\}, \{B,D\}, \{C,D\}, \{A,B,C\}, \{A,B,D\}, \{A,C,D\}, \{B,C,D\}, S\}$$

$$P(\{A\}) = P((0,0.3)) = 0.3 - \tfrac{1}{2}\cdot 0.3^2 = 0.255$$

$$P(\{B\}) = P((0.3,0.6)) = \tfrac{1}{2} + 0.3 - \tfrac{1}{2}\cdot\underbrace{(0.6^2 - 0.3^2)}_{0.27} = 0.665$$

$$P(\{C\}) = P((0.6,0.9)) = 0.3 - \tfrac{1}{2}\underbrace{(0.9^2 - 0.6^2)}_{0.45} = 0.075$$

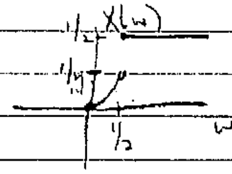
$$P(\{D\}) = P((0.9,1)) = 0.1 - \tfrac{1}{2}\underbrace{(1 - 0.9^2)}_{0.19} = 0.005$$

sum to 1! ✓

$$P(E) = \sum_{x_i \in E} P(\{x_i\}) \qquad x_i \in \{A, B, C, D\}$$

(d)

$$F_X(x) = P(X \le x) \overset{x \le 1/4}{=} P(\omega^2 \le x)$$



$$= \cancel{P(\omega \le \sqrt{x})}$$

$$= P((0, \sqrt{x}))$$

$$F_X(x) = \begin{cases} \sqrt{x} - \tfrac{1}{2}x, & 0 \le x < 1/4 \\ 7/8, & 1/4 \le x < 1/2 \\ 1, & x \ge 1/2 \end{cases}$$

$$\Rightarrow f_X(x) = \begin{cases} 1/2\sqrt{x} - 1/2, & 0 \le x < 1/4 \\ 0, & \end{cases} + \tfrac{1}{2}\delta(x - 1/4) + \tfrac{1}{8}\delta(x - 1/2)$$

(e) $\omega \le 1 \Rightarrow Y \le 5 \Rightarrow P(Y > 6) = 0.$

(b)

$$F_w(x) = P(w \le x)$$

$$\overset{x \ge 0}{=} P((0, x))$$

$$= \begin{cases} x - \frac{1}{2}x^2, & 0 \le x < \frac{1}{2} \\ \frac{1}{2} + x - \frac{1}{2}x^2, & x \ge \frac{1}{2} \\ 0, & x < 0 \\ 1, & x \ge 1 \end{cases}$$

$$f_w(x) = \frac{d}{dx} F_w(x) = \begin{cases} 1 - x, & 0 \le x < 1 \\ 0, & \text{else} \end{cases} + \frac{1}{2}\delta\left(x - \frac{1}{2}\right)$$

$$E[w^2] = \int_0^1 x^2(1-x)\,dx + \int_0^1 x^2 \cdot \frac{1}{2} \cdot \delta\left(x - \frac{1}{2}\right) dx$$

$$= \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^1 + \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2}$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{8} = \frac{8}{24} - \frac{6}{24} + \frac{3}{24} = \frac{5}{24}$$

4)

(a) $P(T \le 1) = P(T \le 1 \mid G)P(G) + P(T \le 1 \mid B)P(B)$

$$= \int_0^1 2e^{-2x}\,dx \cdot 0.9 + \int_0^1 e^{-x}\,dx \cdot 0.1$$

$$= 0.9\left(-e^{-2x}\right)\Big|_0^1 + 0.1\left(-e^{-x}\right)\Big|_0^1$$

$$= 0.9(1-e^{-2}) + 0.1(1-e^{-1})$$

$$= 1 - 0.9e^{-2} - 0.1e^{-1}$$

(b) $P(G \mid T \le 1) = \dfrac{P(T \le 1 \mid G)P(G)}{P(T \le 1)}$

$$= \frac{0.9\cdot(1-e^{-2})}{1-0.9e^{-2}-0.1e^{-1}}$$

$$= \frac{0.9-0.9e^{-2}}{1-0.9e^{-2}-0.1e^{-1}}$$