Final Exam Solutions

ECE 603 Fall, 2006

1)

(a) $\binom{20}{5}(0.1)^5(0.9)^{15}$

(b)

$$\binom{20}{5}(0.1)^5(0.9)^{15}\cdot 0.75$$

$$+\binom{20}{5}(0.5)^5(0.5)^{15}\cdot 0.25$$

(c)

A: event 3 of 1st 5 are defective

$$P(G|A)=\frac{P(A|G)\,P(G)}{P(A)}$$

$$=\frac{\binom{5}{3}(0.1)^3(0.9)^2\cdot 0.75}{\binom{5}{3}(0.1)^3(0.9)^2\cdot 0.75+\binom{5}{3}(0.5)^3(0.5)^2\cdot 0.25}$$

$$P(B|A)=1-P(G|A)$$

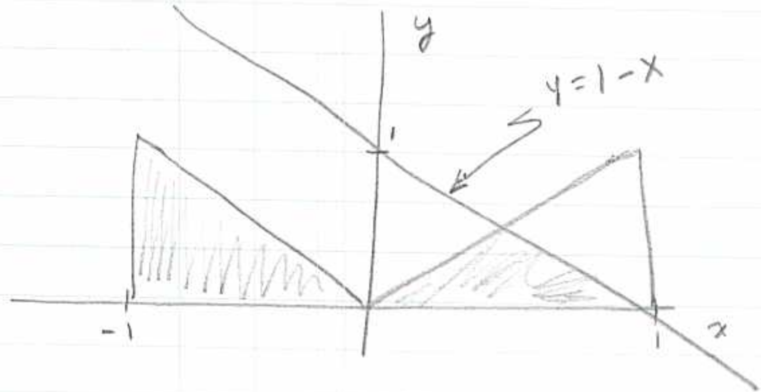
(d)

$P($2 defective in last 15 $|A)$

$$=\binom{15}{2}(0.1)^2(0.9)^{13}\,P(G|A)$$

$$+\binom{15}{2}(0.5)^2(0.5)^{13}\,P(B|A)$$

2)



(a)

$$1 = 2\int_0^1 \int_0^x c x^2 y \, dy \, dx$$

$$= 2\int_0^1 c x^2 y^2/2 \Big|_0^x dx$$

$$= \int_0^1 c x^4 \, dx$$

$$\Rightarrow c = 5$$

(b) $$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_{-1}^{-y} 5x^2 y \, dx + \int_y^1 5x^2 y \, dx$$

$$= 5x^3 y/3 \Big|_{-1}^{-y} + 5x^3 y/3 \Big|_y^1$$

$$= 10 x^3 y/3 \Big|_y^1$$

$$= 10/3 \, (y - y^4) \qquad 0 \le y \le 1$$

Thus, $$f_{X|Y}(x|y) = \frac{5x^2 y}{10/3 \, (y - y^4)}$$

$$= \frac{3/2 \, x^2}{(1 - y^3)} \qquad -1 \le x \le 1$$

For $Y=0.5$,

$$f_{X|Y}(x|0.5) = \frac{3/2\, x^2}{(1-1/8)} = \frac{3/2\, x^2}{7/64} = 12/7\, x^2 \qquad 0.5 \le |x| \le 1$$

(c)

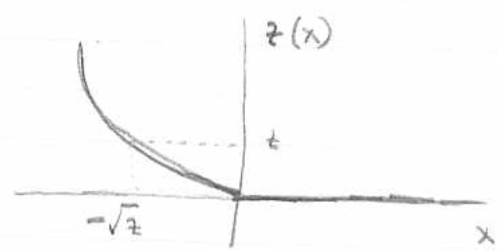
$$P(Y \le 1-x) = 1/2 + \int_0^{1/2}\int_0^{x} 5x^2y\,dy\,dx + \int_{1/2}^{1}\int_0^{1-x} 5x^2y\,dy\,dx$$

(d)

$$f_X(x) = \int_0^x 5x^2y\,dy \qquad 0 \le x \le 1$$

$$= 5/2\, x^2y^2 \Big|_0^x$$

$$= 5/2\, x^4 \qquad \underbrace{-1 \le x \le 1}_{\text{by symmetry}}$$



For $z<0$, $F_Z(z)=0$.

For $z=0$, $F_Z(z)=1/2$.

For $1 \ge z \ge 0$, $F_Z(z) = P(Z \le z) = 1/2 + \int_{-\sqrt{z}}^{0} 5/2\, x^4\,dx$

$$= 1/2 + 1/2\, x^5 \Big|_{-\sqrt{z}}^{0} = 1/2 + 1/2\, z^{5/2}$$

$$f_Z(z) = 1/2\,\delta(z) + \begin{cases} 5/4\, z^{3/2}, & 0 \le z \le 1 \\ 0, & \text{else} \end{cases}$$

3)

(a) By the laws of Large Numbers,

$$E[x] \approx 3.0$$

$\Rightarrow$ (integration by parts) $\lambda = 1/3$

(b)

Have $E[x]$ but also need $Var[x]$.

$$E[x^2] = \int_0^\infty \lambda x^2 e^{-\lambda x}\,dx = \cancel{-x^2 e^{-\lambda x}\Big|_0^\infty}^{\,0} + \int_0^\infty 2x e^{-\lambda x}\,dx$$

$u = x^2 \quad dv = \lambda e^{-\lambda x}$

$du = 2x\,dx \quad v = -e^{-\lambda x}$

$u = 2x \quad dv = e^{-\lambda x}$

$du = 2\,dx \quad v = -1/\lambda\, e^{-\lambda x}$

$$= -2x/\lambda\, e^{-\lambda x}\Big|_0^\infty + \int_0^\infty 2/\lambda\, e^{-\lambda x}\,dx$$

$$= 2/\lambda \left(-1/\lambda\, e^{-\lambda x}\right)\Big|_0^\infty$$

$$= 2/\lambda^2$$

$$Var[x] = 2/\lambda^2 - (1/\lambda)^2 = 1/\lambda^2 = 9.0$$

$$P\left(\sum_{i=1001}^{2000} X_i > 2500\right) = 1 - P\left(\sum_{i=1001}^{2000} X_i < 2500\right)$$

$$= 1 - P\left(\sum_{i=1001}^{2000} X_i - 3000 < 2500 - 3000\right)$$

$$= 1 - P\left(\sqrt{\frac{1}{9000}} \sum_{i=1001}^{2000} (X_i - E(X_i)) < \frac{-500}{30\sqrt{10}}\right)$$

$$= 1 - \left(1/2 - \mathrm{erf}\left(50/3\sqrt{10}\right)\right)$$

$$= 1/2 + \mathrm{erf}\left(50/3\sqrt{10}\right)$$

(c)

$$E[Y^2] = E[(X_{2001} - X_{2002})^2]$$

$$= E[X^2_{2001}] - 2E[X_{2001}]E[X_{2002}] + E[X^2_{2002}]$$

$$= 18 - 2\cdot 3\cdot 3 + 18 = 18$$

$E[Y] = 0$

$\Rightarrow Var[Y] = 18$

5)

(a) $T \sim N(1,4) \Rightarrow -T \sim N(-1,4)$

$\Rightarrow t - T \sim N(t-1, 4)$

(b)

$$E[X(t)] = E[t-T]$$

$$= E[t] - E[T]$$

$$= t - 1$$

$$E[X(t_1)X(t_2)] = E[(t_1-T)(t_2-T)]$$

$$= E[t_1 t_2] - E[t_1 T] - E[t_2 T] + E[T^2]$$

$$= t_1 t_2 - t_1 - t_2 + 5$$

(c) Considering N, $(t_1, t_2, \ldots, t_N)$

$$Z = \sum_{i=1}^{N} \alpha_i X(t_i) = \sum_{i=1}^{N} \alpha_i (t_i - T) = \sum_{i=1}^{N} \alpha_i t_i - \left(\sum_{i=1}^{N} \alpha_i\right) T$$

$$\sim N\left(\sum_{i=1}^{N} \alpha_i t_i - 1,\ 4\left(\sum_{i=1}^{N} \alpha_i\right)^2\right)$$

$\Rightarrow X(t_1), X(t_2), \ldots, X(t_N)$ are jointly Gaussian $\Rightarrow X(t)$ is a Gaussian RP

4)

It converges in every way. Show m.s. and get the rest by implication:

$$E\left[\frac{1}{N}\sum_{i=1}^{N} X_i\right] = \mu$$

$$\begin{aligned} \mathrm{Var}\left[\frac{1}{N}\sum_{i=1}^{N} X_i\right] &= \frac{1}{N^2}\sum_{i=1}^{N}\mathrm{Var}[X_i] \\ &= \frac{1}{N^2}\cdot N \cdot \left(E[X^2] - \mu^2\right) \\ &= \frac{c}{N} \end{aligned}$$

Thus

$$E\left[\left(\frac{1}{N}\sum_{i=1}^{N} X_i - \mu\right)^2\right] = \frac{c}{N} \to 0$$

$$\Rightarrow \frac{1}{N}\sum_{i=1}^{N} X_i \xrightarrow{m.s.} \mu$$

$$\Rightarrow \frac{1}{N}\sum_{i=1}^{N} X_i \xrightarrow{P} \mu$$

$$\Rightarrow \frac{1}{N}\sum_{i=1}^{N} X_i \xrightarrow{D} \mu$$