Final Exam Solutions

ECE 603 Fall, 2016

1) Let $\underset{\wedge}{B}_i$: event of "big" on X_i

(a) $P(B_1) = P(B_1|H)P(H) + P(B_1|T)P(T)$

$= \frac{1}{2}\cdot\frac{1}{2} + \frac{3}{4}\cdot\frac{1}{2}$

$= \frac{5}{8}$

(b) $P(B_2|B_1) = \dfrac{P(B_1 \cap B_2)}{P(B_1)}$

$= \dfrac{P(B_1 \cap B_2|H)P(H) + P(B_1 \cap B_2|T)P(T)}{P(B_1)}$

$= \dfrac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{2}}{\frac{5}{8}}$

$= \dfrac{\frac{1}{8} + \frac{9}{32}}{\frac{5}{8}} = \dfrac{\frac{13}{32}}{\frac{20}{32}} = \frac{13}{20}$

(c) $P(\underbrace{\text{15 are "big" out of 20}}_{\text{call this } A})$

$= P(A|H)P(H) + P(A|T)P(T)$

$= \binom{20}{15}\left(\frac{1}{2}\right)^{15}\cdot\left(\frac{1}{2}\right)^{5}\cdot\frac{1}{2} + \binom{20}{15}\left(\frac{3}{4}\right)^{15}\cdot\left(\frac{1}{4}\right)^{5}\cdot\frac{1}{2}$

(d) $P(T|A) = \dfrac{P(A|T)P(T)}{P(A)} = \dfrac{\binom{20}{15}\left(\frac{3}{4}\right)^{15}\cdot\left(\frac{1}{4}\right)^{5}\cdot\frac{1}{2}}{(\cdot)}$

(e) C: more than 750 are big

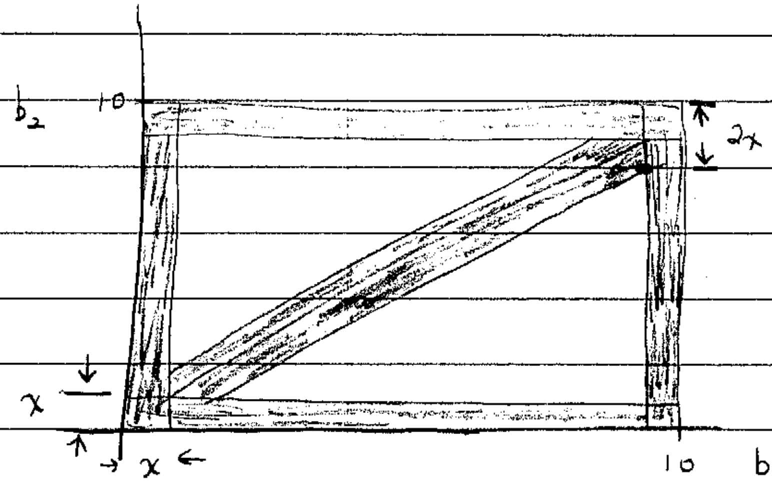
$P(C) = \cancel{P(C|H)}^{\approx 0}P(H) + \cancel{P(C|T)}^{1/2}P(T)$ (or could use CLT)

$= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

2)

First, find the CDF $F_X(x)$.

Key is the picture for $P(X \le x)$.



$$F_X(x) = \frac{100 - 2 \cdot \frac{1}{2}(10-3x)(10-3x)}{100}, \quad 0 \le x \le 10/3$$

$$= 1 - \frac{1}{100}(100 - 60x + 9x^2)$$

$$= 3/5\, x - 9/100\, x^2$$

$$f_X(x) = d/dx\, F_X(x) = \begin{cases} 3/5 - 9/50\, x, & 0 \le x \le 10/3 \\ 0, & \text{else} \end{cases}$$

3)

(a)

$$\int_0^\infty \int_0^\infty c e^{-(\lambda x+\mu y)} dx\,dy = c\int_0^\infty e^{-\lambda x}dx \int_0^\infty e^{-\mu y}dy$$

$$= c\left(-\tfrac{1}{\lambda} e^{-\lambda x}\right)\Big|_0^\infty \left(-\tfrac{1}{\mu} e^{-\mu y}\right)\Big|_0^\infty$$

$$= c/\lambda\mu = 1 \Rightarrow c = \lambda\cdot\mu$$

(b) $$f_X(x) = \int_0^\infty \lambda\cdot\mu\cdot e^{-(\lambda x+\mu y)}dy = \lambda e^{-\lambda x} \cancelto{1}{\int_0^\infty \mu e^{-\mu y}dy}$$

$$= \lambda e^{-\lambda x}, \quad x \geq 0$$

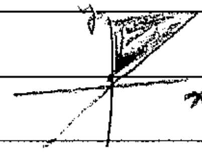
$$f_Y(y) = \int_0^\infty \lambda\cdot\mu\cdot e^{-(\lambda x+\mu y)}dx = \mu e^{-\mu y} \cancelto{1}{\int_0^\infty \lambda e^{-\lambda x}dx}$$

$$= \mu e^{-\mu y}, \quad y \geq 0$$

(c) $$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\lambda\mu e^{-(\lambda x+\mu y)}}{\lambda e^{-\lambda x}} = \mu e^{-\mu y}, \quad \begin{matrix} 0 \leq x \leq \infty \\ 0 \leq y \leq \infty \end{matrix}$$

(d) <u>Yes.</u> $f_{Y|X}(y|x) = f_Y(y)$

(e)



$$P(Y>X) = \int_0^\infty \int_x^\infty \lambda\mu e^{-(\lambda x+\mu y)}dy\,dx$$

$$= \int_0^\infty \lambda\mu e^{-\lambda x}\left(\tfrac{1}{\mu} e^{-\mu y}\right)\Big|_{y=x}^\infty dx$$

$$= \int_0^\infty \lambda e^{-\lambda x} e^{-\mu x}dx$$

$$= \frac{-\lambda}{\lambda+\mu} e^{-(\lambda+\mu)x}\Big|_0^\infty = \lambda/_{\lambda+\mu}$$

4)

(a) Consider $R_X(t_1, t_1) = 4t_1$. Does not just depend on $|t_2 - t_1| = 0$.

$\underline{\underline{\text{not}}}$ WSS ; thus, also $\underline{\underline{\text{not}}}$ SSS

(b) $P(t) = E[X(t)X(t)] = R_X(t,t) = 4t$

(c)

$X(2) \sim N(0, 4\cdot 2) = N(0, 8)$

$\frac{x-\mu}{\sqrt{2}\sigma} = \frac{1}{\sqrt{2}\cdot\sqrt{8}}$

$P(X(2) > 1) = 1 - P(X(2) \leq 1) = 1 - (1/2 + 1/2\, \mathrm{erf}(1/4))$

$= 1/2 - 1/2\, \mathrm{erf}(1/4)$

(d) I claim $V_n \xrightarrow{m.s.} 0$

$E[|V_n - 0|^2] = E[|V_n|^2] = 4 \cdot 1/n = 4/n \to 0$

Hence, $V_n \xrightarrow{m.s.} 0$ and thus $V_n \xrightarrow{P} 0$, $V_n \xrightarrow{D} 0$.

(e)

$m_Z[n] = E[Z_n] = E[Y_n - Y_{n-1}] = \cancelto{0}{m_X(n)} - \cancelto{0}{m_X(n-1)} = 0$

$R_Z[m,n] = E[Z_m Z_n] = E[(Y_m - Y_{m-1})(Y_n - Y_{n-1})]$

$= E[X(m)X(n)] + E[X(m-1)X(n-1)] - E[X(m)X(n-1)]$

$- E[X(m-1)X(n)]$

$$= 4\cdot\min(m,n) + 4\cdot\min(m-1,n-1)$$
$$- 4\cdot\min(m,n-1) - 4\cdot\min(m-1,n)$$

<u>Suppose $m \le n-1$</u>

$$R_z[m,n] = 4m + 4(m-1) - 4m - 4\cdot(m-1) = 0$$

<u>Suppose $m=n$</u>

$$R_z[n,n] = 4m + 4\cdot(m-1) - 4\cdot(m-1) - 4\cdot(m-1)$$
$$\cancel{= 4m + 4m - 4 - 4m + 4 - 4m + 4}$$
$$= 4$$

<u>Suppose $m \ge n+1$</u>

$$R_z[m,n] = 4\cdot n + 4\cdot(n-1) - 4\cdot(n-1) - 4n$$
$$= 0$$

$$\Rightarrow R_z[m,n] = \begin{cases} 4, & |m-n| = 0 \\ 0, & \text{else} \end{cases}$$

(f) <u>Yes</u>

$m_z(n) = 0$ constant

$R_z[m,n]$ only depends on $|m-n|$.