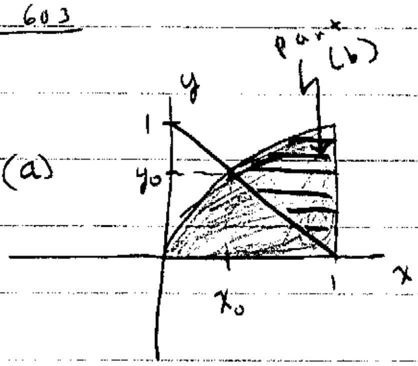
Final Exam Solutions

ECE 603 Fall, 2015

1) (a)



$$\int_0^1 \int_{y^2}^1 cy\,dx\,dy = c\int_0^1 yx \Big|_{y^2}^1 dy$$
$$= c\int_0^1 (y - y^3)\,dy$$
$$= c\left(\tfrac{1}{2}y^2 - \tfrac{1}{4}y^4\right)\Big|_0^1$$
$$= c/4$$
$$\Rightarrow c = 4$$

(b) Need to find x_0:

$$y_0 = 1 - x_0 \quad \text{and} \quad x_0 = y_0^2$$

$$\Rightarrow y_0^2 + y_0 - 1 = 0 \Rightarrow y_0 = \frac{-1 \pm \sqrt{5}}{2} \quad \text{use } -1/2 + \sqrt{5}/2 \in (0,1)$$

$$x_0 = \left(-1/2 + \sqrt{5}/2\right)^2$$

and

$$P(Y > 1 - X) = \int_{x_0}^1 \int_{1-x}^{\sqrt{x}} 4y\,dy\,dx$$

(c) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

$$0 \le x \le 1 \quad \Downarrow \quad = \int_0^{\sqrt{x}} 4y\,dy = 2y^2\Big|_0^{\sqrt{x}} = 2x$$

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$ integrates to 1! ✓

(d) $$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} 4y/2x = 2y/x, & 0 \le x \le 1,\ 0 \le y \le \sqrt{x} \\ 0, & \text{else} \end{cases}$$

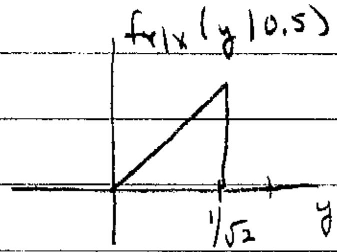
Note: Integrates to 1 for any x! ✓

(e)

No. $f_{Y|X}(y|x) \neq f_Y(y)$ (it depends on x!)

(f)

$$f_{Y|X}(y|0.5) = \begin{cases} 4y, & 0 \le y \le 1/\sqrt{2} \\ 0, & \text{else} \end{cases}$$



ML estimate $= 1/\sqrt{2}$ (maximum of $f_{Y|X}(y|0.5)$ is here)

MMSE estimate $= E[Y|X=0.5] = \int_0^{1/\sqrt{2}} 4y^2\,dy = \frac{4}{3}y^3\Big|_0^{1/\sqrt{2}} = \frac{2}{3}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{3}$

2) Let G_i: event Alice scores in game i

W_i: event team wins game i

(a)

$$P(G_2) = P(G_2|G_1)P(G_1) + P(G_2|\bar{G}_1)\cdot P(\bar{G}_1)$$
$$= 0.8 \cdot 0.6 + 0.5 \cdot 0.4$$
$$= 0.68$$

(b) $$P(G_1|G_2) = \frac{P(G_2|G_1)P(G_1)}{P(G_2)}$$
$$= \frac{0.8 \cdot 0.6}{0.68} = \frac{0.48}{0.68} = \frac{48}{68} = 12/17$$

(c) $$P(G_3|G_1) = P(G_3 \cap (G_2 \cup \bar{G}_2)|G_1)$$

since $G_2 \cup \bar{G}_2 = S$

$$= P(G_3 \cap G_2 | G_1) + P(G_3 \cap \bar{G}_2 | G_1)$$
$$= P(G_3 | G_2, G_1) P(G_2|G_1) + P(G_3|\bar{G}_2, G_1)P(\bar{G}_2|G_1)$$
$$= P(G_3|G_2)P(G_2|G_1) + P(G_3|\bar{G}_2)P(\bar{G}_2|G_1)$$
$$= 0.8 \cdot 0.8 + 0.5 \cdot 0.2$$
$$= 0.74$$

(many other approaches, also!)

3)

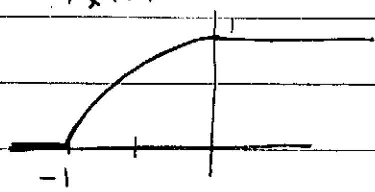
(a) $P(X_0 \geq -1/2) = 1 - P(X_0 < -1/2)$

$= 1 - F_{X_0}(-1/2)$

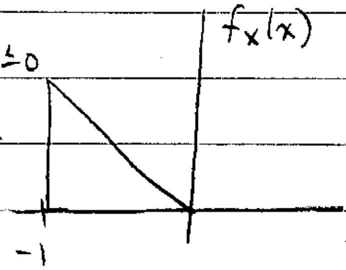
$= 1 - (1 - (-1/2)^2)$

$= 1/4$

$F_x(x) =$



(b) $f_x(x) = d/dx\, F_x(x) = \begin{cases} -2x, & -1 \leq x \leq 0 \\ 0, & \text{else} \end{cases}$



$E[x] = \int_{-1}^{0} x(-2x)\,dx$

$= -2/3\, x^3 \Big|_{-1}^{0}$

$= -2/3$

(c) $E[x^2] = \int_{-1}^{0} x^2(-2x)\,dx = -2/4\, x^4 \Big|_{-1}^{0} = 1/2$

$Var(X) = 1/2 - (2/3)^2 = 1/18$

(d) $P(U \geq -115) = P(U - ((-2/3)\cdot 180) \geq -115 - (-2/3 \cdot 180))$

$= P(U - 180\mu \geq 5)$

$= P\left(\frac{U - 120\mu}{\sqrt{180 \cdot 1/18}} \geq 5/\sqrt{10}\right)$

$= P(Z \geq 5/\sqrt{10})$

$= 1 - P(Z \leq 5/\sqrt{10})$

$= 1 - (1/2 + erf(5/\sqrt{10})) = 1/2 - erf(5/\sqrt{10})$

(e)

$$m_Y[n] = E[Y_n]$$
$$= E[X_n - X_{n-1}]$$
$$= E[X_n] - E[X_{n-1}]$$
$$= -2/3 - (-2/3)$$
$$= 0$$

(f)

$$R_Y[m,n] = E[Y_m Y_n]$$
$$= E[(X_m - X_{m-1})(X_n - X_{n-1})]$$
$$= E[X_m X_n] - E[X_{m-1} X_n] - E[X_{n-1} X_m] + E[X_{n-1} X_{m-1}]$$

If $|m-n|>1$,

$$R_Y[m,n] = (-2/3)^2 - (-2/3)^2 - (-2/3)^2 + (-2/3)^2 = 0$$

If $m=n$

$$R_Y[m,n] = 1/2 - (-2/3)^2 - (-2/3)^2 + 1/2$$
$$= 1/9$$

If $|m-n|=1$

$$R_Y[m,n] = (-2/3)^2 - (1/2)^2 - (2/3)^2 + (2/3)^2$$
$$= -1/18$$

$$R_Y[m,n] = \begin{cases} 1/9, & m=n \\ -1/18, & |m-n|=1 \\ 0, & \text{else} \end{cases}$$

yes. WSS!

4) (a)

$$P_X = R_X(0) = 20$$

(b)

$$m_Y(t) = E\left[4\cos(2\pi 30 t) + X(t)\right]$$

$$= 4\cos(2\pi 30 t) + \cancelto{0}{E[X(t)]}$$

$$= 4\cos(2\pi 30 t)$$

$$R_Y(t_1, t_2) = E\left[Y(t_1)Y(t_2)\right]$$

$$= E\left[\left(4\cos(2\pi 30 t_1) + X(t_1)\right)\left(4\cos(2\pi 30 t_2) + X(t_2)\right)\right]$$

$$= 16\cos(2\pi 30 t_1)\cos(2\pi 30 t_2) + 4\cos(2\pi 30 t_1)\cancelto{0}{E[X(t_1)]} + 4\cos(2\pi 30 t_2)\cancelto{0}{E[X(t_2)]} + E\left[X(t_1)X(t_2)\right]$$

$$= 16\cos(2\pi 30 t_1)\cos(2\pi 30 t_2) + 20\,\mathrm{sinc}^2(80(t_2 - t_1))$$

$m_Y(t)$ <u>not</u> constant, <u>not</u> WSS.

(c)

<u>Yes</u>

Consider any $Y(t_1), Y(t_2), \ldots, Y(t_N)$

$$Z = \sum_{i=1}^{N} a_i Y(t_i) = \underbrace{4\sum_{i=1}^{N} a_i \cos(2\pi t_i)}_{\text{constant}} + \underbrace{\sum_{i=1}^{N} a_i X(t_i)}_{\text{Gaussian - since } X(t_1), X(t_2), \ldots, X(t_N) \text{ jointly Gaussian}}$$

$\Rightarrow Z$ is Gaussian

$\Rightarrow$ by Cramer-Wald Device, $Y(t_1), Y(t_2), \ldots, Y(t_N)$ are jointly Gaussian

$\Rightarrow Y(t)$ is a Gaussian RP

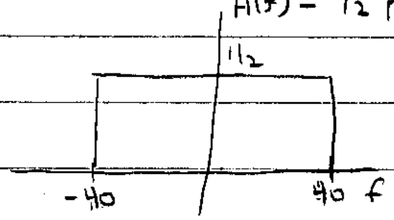
(d)

$$z(t) = h(t) * Y(t)$$

$$= h(t) * (4\cos(2\pi 30t) + X(t))$$

$$= h(t) * 4\cos(2\pi 30t) + h(t) * X(t)$$

Now, $H(f) = \frac{1}{2}\, p(f/80)$



$$\Rightarrow \quad z(t) = 2\cos(2\pi 30t) + u(t) \qquad u(t) \triangleq h(t) * X(t)$$

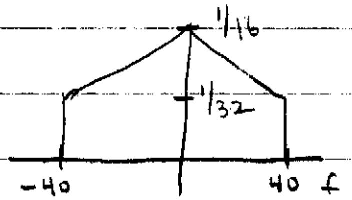
$$P_z = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E\left[(2\cos(2\pi 30t) + u(t))^2\right] dt$$

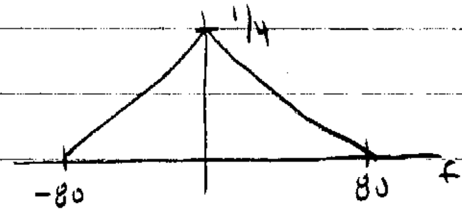
$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(4\cos^2(2\pi 30t) + 4\cos(2\pi 30t)\, \cancel{E[u(t)]}^{\,0} + E[u^2(t)]\right) dt$$

$$= \lim_{T\to\infty} \frac{4}{2T} \int_{-T}^{T} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(2\pi 60t)\right) dt + E[u^2(t)]$$

$$= 2 + E[u^2(t)]$$

Now, $S_u(f) = |H(f)|^2 S_x(f)$ where $S_x(f) = \frac{1}{4}\Lambda(f/80)$





$$P_u = \int_{-\infty}^{\infty} S_u(f)\, df = 80 \cdot \tfrac{1}{32} + 80 \cdot \tfrac{1}{32} \cdot \tfrac{1}{2}$$

$$= \tfrac{5}{2} + \tfrac{5}{4} = \tfrac{15}{4}$$

$$\Rightarrow P_z = \tfrac{23}{4}$$