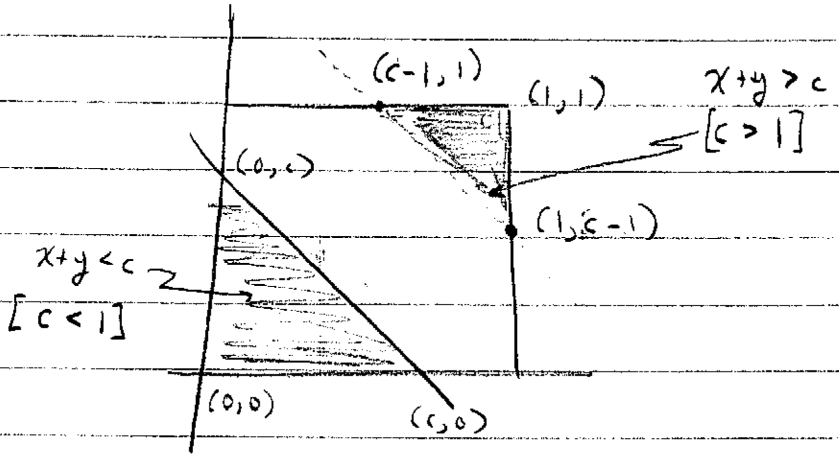
1)



$S = [0,2]$

$\mathcal{A} = \mathbb{B}$ (restricted to $[0,2]$, of course)

$$P(\omega < c) = \begin{cases} c \cdot c/2, & 0 \le c \le 1 \\ 1 - \frac{(2-c)(2-c)}{2}, & 2 \ge c > 1 \\ 0, & c < 0 \\ 1, & c > 2 \end{cases}$$

$$= \begin{cases} 0, & c \le 0 \\ c^2/2, & 0 < c \le 1 \\ 2c - 1 - c^2/2, & 1 < c \le 2 \\ 1, & c > 2 \end{cases}$$

and

$$P((a,b)) = \begin{cases} (b^2 - a^2)/2, & 0 \le a < b \le 1 \\ 2b - 1 - b^2/2 - a^2/2, & 0 \le a \le 1 < b \le 2 \\ \underbrace{(2b - 1 - b^2/2) - (2a - 1 - a^2/2)}_{= 2(b-a) - (b^2 - a^2)/2}, & 1 < a < b < 2 \end{cases}$$

2)

(a)

Consider any $x \in [0,1]$. Write its binary expansion:

$$0.b_1 b_2 b_3 \ldots$$

For any such x, $\exists\, \omega = (b_1, b_2, b_3, \ldots) \in S$

$\Rightarrow S$ uncountable

(b)

I claim $Y_n \to 0$ a.s., $\xrightarrow{P}$, $\xrightarrow{D}$

If $\omega_i = 0$ for any i, then

$$\lim_{n \to \infty} Y_n(\omega) = 0.$$

But the set of ω s.th. $\omega_i = 0$ for some i has $P(A) = 1$. Why? Consider the set A_n such that $\omega_i = 0$ for some $i \le n$. Then, $P(A_n) = 1 - 1/2^n$ and $\lim_{n \to \infty} P(A_n) = \underline{1}$.

But it does not go in m.s.

$$E[|Y_n - 0|^2] = 1/2^n |(1.5)^n|^2 + (1 - 1/2^n) \cdot 0$$

$$= (2.25/2)^n \to \infty$$

(c) I claim it converges in distribution to Z s.th. $P(Z=0) = 1/2$, $P(Z=1) = 1/2$. Note that $P(Z_n = 0) = 1/2$, $P(Z_n = 1) = 1/2$ for all n!

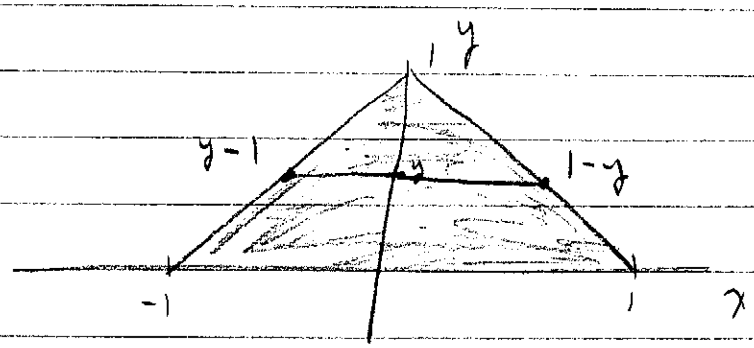
But it does not converge any other way; in particular; for any

$$P\left(\{\omega : |X_n(\omega) - X(\omega)| > 1/2\}\right) \geq 1/2$$

for either n or $n+1$ (for any n); thus, X_n does not converge in probability

$\Rightarrow X_n$ does not converge a.s., m.s.

3)



(a) $\frac{1}{2} \cdot 2 \cdot 1 \cdot c = 1 \implies c = 1$

(b) $f_Y(y) = \int_{y-1}^{1-y} 1 \, dx = 1-y-(y-1)$

$= 2-2y, \quad 0 \le y \le 1$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \begin{cases} \dfrac{1}{2(1-y)}, & 0 \le y \le 1 \\ & y-1 \le x \le 1-y \\ 0, & \text{else} \end{cases}$$

(c) $f_{X|Y}(x|1/2) = 1 \qquad -1/2 \le x \le 1/2$

$$P(X > 1/4 \mid Y = 1/2) = \int_{1/4}^{\infty} f_{X|Y}(x|1/2) \, dx$$

$$= 1/4$$

5) (a) $P(Y > 3) = P(X_1^2 + 2 > 3)$

$$= P(X_1^2 > 1)$$

$$= P(X_1 \le -1) + P(X_1 \ge 1)$$

$$= P(X_1 \le -1) + 1 - P(X_1 \le 1)$$

$\left(X_1 \sim N(0, R_Z(0)) = N(0, 1/2)\right)$

$$\frac{-1-0}{\sqrt{1/2}} \quad \downarrow \quad = \frac{1}{2} - \text{erf}(\sqrt{2}) + 1 - \left(\frac{1}{2} + \text{erf}(\sqrt{2})\right)$$

$$\frac{1-0}{\sqrt{1/2}} \quad = 1 - 2\,\text{erf}(\sqrt{2})$$

(b) X_2 is a linear combination of jointly Gaussian rvs $\Rightarrow$ Gaussian

$$\mu_{X_2} = E[X_2] = \frac{1}{2} E[Z(0)] + \frac{1}{2} E[Z(1)] = 0$$

$$\sigma_{X_2}^2 = \text{Var}[X_2] = E[X_2^2] - \cancel{(E[X_2])^2}^{\,0}$$

$$= \frac{1}{4} E[Z^2(0)] + \frac{1}{2} E[Z(0)Z(1)] + \frac{1}{4} E[Z^2(1)]$$

$$= 1/4 \cdot 1/2 + \frac{1}{2} \cdot \frac{1}{2} e^{-1/4} + 1/4 \cdot 1/2$$

$$= 1/4 + 1/4\, e^{-1/4}$$

$$f_{X_2}(x) = \frac{1}{\sqrt{2\pi\sigma_{X_2}^2}} e^{-x^2/2\sigma_{X_2}^2}$$

$$P(X_2 > 1) = 1 - P(X_2 \le 1) = 1 - \text{erf}\left(1/\sqrt{1/4 + 1/4\, e^{-1/4}}\right) \quad \left(\frac{1-0}{\sqrt{1/4 + 1/4\, e^{-1/4}}}\right)$$

$$= 1 - \text{erf}\left(2/\sqrt{1 + e^{-1/4}}\right)$$