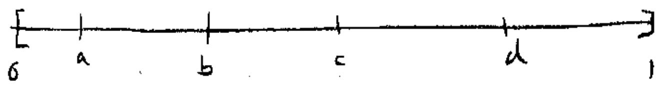
Final Exam Solutions

ECE 603 — Fall, 2008

1) (a)



First note that any singleton is in $\mathbb{B}$:

$$\{x\} = \bigcap_{n=1}^{\infty} \left(x - 1/n,\ x + 1/n\right)$$

and thus so are the rationals (because they are countable)

$$\Rightarrow \underbrace{(a,b) \cap \mathbb{Q}}_{A \triangleq} \in \mathbb{B}$$

So is $\overline{\mathbb{Q}}$ (complement of $\mathbb{Q}$); thus

$$\Rightarrow \underbrace{(c,d) \cap \overline{\mathbb{Q}}}_{B \triangleq} \in \mathbb{B}$$

$$\Rightarrow [0,1] \cap \overline{A} \cap \overline{B} \in \mathbb{B}$$

(b)

$$P\big(S(a,b,c,d)\big) = (d-a) - P(A) - P(B)$$

$$= (d-a) - (d-c)$$

$$= c - a$$

(c)

$$P\big(S(0, 1/4, 1/2, 1)\big) = 1 - P(A) - P(B)$$

$$= 1 - 0 - \frac{1^2 - 1/2^2}{2}$$

$$= 5/8$$

2) (a)

Let c: # of bad chips ≤ 3

$$P(c \mid X) = 0.5$$
$$P(c \mid Y) = 0.85$$

$$P(c) = P(c \mid X)P(X) + P(c \mid Y)P(Y)$$
$$= 0.5 \cdot 0.25 + 0.85 \cdot 0.75$$
$$= 1/8 + 17/20 \cdot 3/4$$
$$= 61/80$$

(b) $$P(Y \mid c) = \frac{P(c \mid Y)P(Y)}{P(c)}$$
$$= \frac{17/20 \cdot 3/4}{61/80}$$
$$= 51/61$$

$$P(X \mid c) = 10/61$$

(c) $$P(c_2 \mid c_1) = \frac{P(c_1 \cap c_2)}{P(c_1)} = \frac{P(c_1 \cap c_2 \mid X)P(X) + P(c_1 \cap c_2 \mid Y)P(Y)}{P(c_1)}$$
$$= \frac{1/\cancel{16} + 17/20 \cdot 17/\cancel{20} \cdot 3/\cancel{4}}{61/\cancel{80}}$$
$$= \frac{967}{1220}$$

(d) $$P(c \text{ on at least } 5) = P(c \text{ on at least } 5 \mid X)P(X) + P(c \text{ on at least } 5 \mid Y)P(Y)$$
$$= \sum_{k=5}^{10} \binom{10}{k} 0.5^k \, 0.5^{10-k} \cdot 0.25 + \sum_{k=5}^{10} \binom{10}{k} 0.85^k \, 0.15^{10-k} \cdot 0.75$$

3)

(a)

Y is uniform on [-2,1] (flip pdf of X around x=0).

$$\Rightarrow f_Y(y) = \begin{cases} 1/3, & -2 \le y \le 1 \\ 0, & \text{else} \end{cases}$$

(b)

$$\int_0^m 2(1-x)\,dx = 1/2$$

$$\left(x - \tfrac{1}{2}x^2\right)\Big|_0^m = 1/4$$

$$m - \tfrac{1}{2}m^2 = 1/4$$

$$2m^2 - 4m + 1 = 0$$

$$m = \frac{4 \pm \sqrt{16-8}}{4} = 1 \pm \sqrt{2}/2 \Rightarrow \text{must be } m = 1 - 1/\sqrt{2}$$

(c)

$$d/dx_0\, E[(X-x_0)^2] = d/dx_0 \int_{-\infty}^{\infty} (x-x_0)^2 f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} \cancel{-2}(x-x_0)\, f_X(x)\,dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} x\, f_X(x)\,dx - x_0 \int_{-\infty}^{\infty} f_X(x)\,dx = 0$$

$$\Rightarrow x_0 = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$$

$$= E[X]$$

4) $S_x(f) = 10$

(a) $S_y(f) = \begin{cases} 10, & |f| < 3 \\ 0, & \text{else} \end{cases}$

$\Rightarrow P_y = \int_{-\infty}^{\infty} S_y(f)\,df = 60$

(b) $S_z(f) = \begin{cases} 10e^{-f^2/2}, & |f| < 3 \\ 0, & \text{else} \end{cases}$

$P_z = \int_{-3}^{3} 10e^{-f^2/2}\,df$

$= 10 \cdot \sqrt{2\pi} \int_{-3}^{3} \frac{1}{\sqrt{2\pi}} e^{-f^2/2}\,df$

$= 10 \cdot \sqrt{2\pi} \left(P(Z \le 3) - P(Z \le -3)\right)$ $\qquad Z \sim N(0,1)$

$= 10 \cdot \sqrt{2\pi} \left(\cancel{\tfrac{1}{2}} + \operatorname{erf}(3) - (\cancel{\tfrac{1}{2}} - \operatorname{erf}(3))\right)$

$= 20 \cdot \sqrt{2\pi}\, \operatorname{erf}(3)$

5)

(a)

• $m_Y(t) = E[Y(t)] = E[t^2 + X(t)]$

$$= t^2$$

$$R_Y(t_1, t_2) = E[(t_1^2 + X(t_1))(t_2^2 + X(t_2))]$$

$$= t_1^2 t_2^2 + t_1^2 \cancel{E[X(t_2)]}^{\,0} + t_2^2 \cdot \cancel{E[X(t_1)]}^{\,0}$$

$$+ E[X(t_1)X(t_2)]$$

$$= t_1^2 t_2^2 + R_X(t_2 - t_1)$$

• not WSS

• $Y(2) \sim N(4, R_X(0))$

(b)

• $m_Z(t) = E[Z(t)]$

$$= E[X^2(t)] - R_X(0)$$

$$= 0$$

$$R_Z(t_1, t_2) = E[Z(t_1)Z(t_2)]$$

$$= E[(X^2(t_1) - R_X(0))(X^2(t_2) - R_X(0))]$$

$$= E[X^2(t_1)X^2(t_2)] + \cancel{R_X^2(0) - 2R_X^2(0)} - R_X^2(0)$$

$$= \cancel{E[X^2(t_1)]E[X^2(t_2)]} + E[X(t_1)X(t_2)]E[X(t_1)X(t_2)]$$

$$+ E[X(t_1)X(t_2)]E[X(t_1)X(t_2)] \cancel{- R_X^2(0)}$$

$$= 2R_X^2(t_2 - t_1)$$

$$= 2R_X^2(\tau)$$

• $Z(t)$ is WSS and

$$S_Z(f) = 2\, S_X(f) * S_X(f)$$

• $Z(2) = X^2 - R_X(0)$

where $X \sim N(0, R_X(0))$

Ignore $R_X(0)$ for now:

$$F_{\tilde{Z}(2)}(z) = P(Z \le z)$$

$$= P(X^2 \le z)$$

$$= P(-\sqrt{z} \le X \le \sqrt{z})$$

$$= \int_{-\sqrt{z}}^{\sqrt{z}} 1/\sqrt{2\pi R_X(0)}\, e^{-x^2/2R_X(0)}\, dx$$

$$f_{\tilde{Z}}(z) = d/dz\, F_{\tilde{Z}}(z)$$

$$= 1/\sqrt{2\pi R_X(0)}\, e^{-z/2R_X(0)} \cdot 1/2\, z^{-1/2}$$

$$+ 1/\sqrt{2\pi R_X(0)}\, e^{-z/2R_X(0)} \cdot (+1/2\, z^{-1/2})$$

$$= 1/\sqrt{2\pi R_X(0) \cdot z}\, e^{-z/2R_X(0)} \qquad z \ge 0$$

and now shift the pdf

$$f_{Z(2)}(z) = \begin{cases} 1/\sqrt{2\pi R_X(0)\,(z + R_X(0))}\; e^{-(z+R_X(0))/2R_X(0)} & -R_X(0) \le z < \infty \\ 0, & \text{else} \end{cases}$$