1)

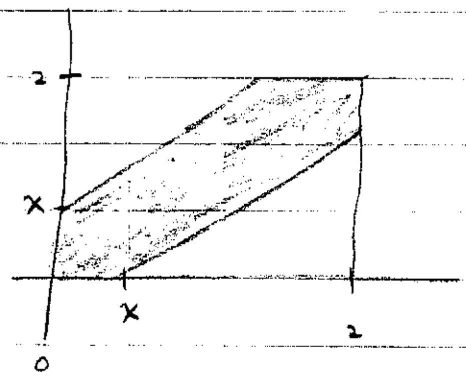
The distance is between 0 and 2. Thus,

$$\Omega = [0,2]$$

and

$$\mathcal{A} = \text{Borel set restricted to } [0,2]$$

Need to find distribution (or density function)




$$P(X \le x) = \frac{\text{Area of strip}}{\text{Total Area}} = \frac{4-(2-x)^2}{4} = \frac{4x-x^2}{4} = x - \tfrac{1}{4}x^2$$

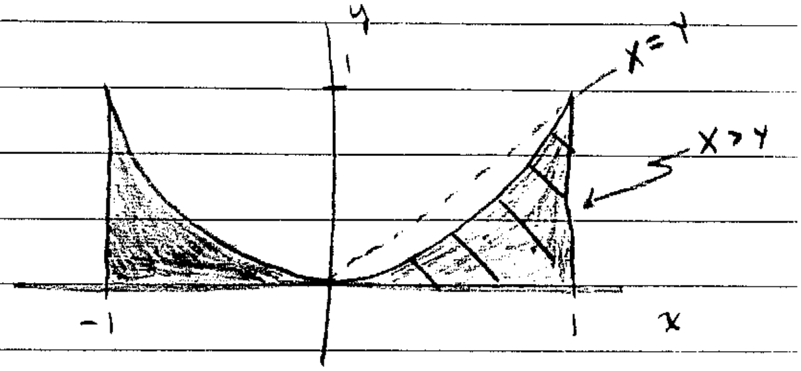
Thus, $F_X(x) = \begin{cases} x - \tfrac{1}{4}x^2, & 0 \le x \le 2 \\ 0, & \text{else} \end{cases}$

and

$$P((a,b)) = b - \tfrac{1}{4}b^2 - (a - \tfrac{1}{4}a^2), \quad 0 \le a \le b \le 2$$
$$= (b-a) - \tfrac{1}{4}(b^2 - a^2)$$

All other sets in $\mathcal{A}$ are generated from above.

2)




(a)

$$\int_{-1}^{1}\int_{0}^{x^2} cy\, dy\, dx = \int_{-1}^{1} \tfrac{1}{2} c y^2 \Big|_0^{x^2} dx$$

$$= \int_{-1}^{1} \tfrac{1}{2} c x^4 \, dx = \tfrac{1}{10} c x^5 \Big|_{-1}^{1}$$

$$= \tfrac{1}{5} c \implies c = 5$$

(b)

$$f_X(x) = \int_0^{x^2} 5y\, dy = \tfrac{5}{2} y^2 \Big|_0^{x^2} = \tfrac{5}{2} x^4, \quad -1 \le x \le 1$$

$$= \begin{cases} \tfrac{5}{2} x^4, & -1 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

(c)

$P(X > Y) = 1/2$ (see picture)

(Note this requires symmetry about $x=0$ not only of the picture but also the function.)

(d)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_{-1}^{-\sqrt{y}} 5y\,dx + \int_{\sqrt{y}}^{1} 5y\,dx \qquad , 0 \le y \le 1$$

$$= 5xy \Big|_{-1}^{-\sqrt{y}} + 5xy \Big|_{\sqrt{y}}^{1}$$

$$= 5y - 5y^{3/2} + 5 - 5y^{3/2}$$

$$= 10y - 10y^{3/2}$$

$$= \begin{cases} 10y - 10y^{3/2}, & 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$$

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{2(1-y^{1/2})}, & 0 \le y \le 1,\ y \le x^2 \le 1 \\ 0, & \text{else} \end{cases}$$

3) $\sigma_X^2 = E[X^2] - (E[X])^2 = 4$

(a) Chebyshev's $\neq$

$$P(|X - E[X]| \ge \varepsilon) \le \frac{\sigma_X^2}{\varepsilon^2}$$

$$P(|X - 2| \ge 10) \le \frac{4}{100} = 1/25$$

(b)

Simply apply erf(·)

$$P(X \geq 10) = 1 - P(X < 12)$$

$$= 1 - \left( \frac{1}{2} + \text{erf}\left( \frac{12-2}{2} \right) \right)$$

$$= \frac{1}{2} - \text{erf}(5)$$

4)

Clearly, the limiting random variable is 0 in some sense:

$$P(|X_n - 0| \geq \varepsilon) = 2^{-n} \to 0 \quad \text{as } n \to \infty \text{ for all } \varepsilon > 0.$$

Thus $X_n \xrightarrow{P} 0 \Rightarrow X_n \xrightarrow{D} 0$

Let's try mean square:

$$E[|X_n - 0|^2] = E[|X_n|^2] = 2^{-n} 2^{2n} = 2^n \to \infty \quad \text{as } n \to \infty$$

Thus, $n \not\xrightarrow{m.s.} 0$

We cannot tell about a.s. since $X_n(\omega)$ is not given explicitly.

5)

(a) No.

$$R_X(\tau) = p(\tau) \Longleftrightarrow S_X(f) = \mathrm{sinc}(f)$$

but $\mathrm{sinc}(f)$ can be negative, which violates $S_X(f) \geq 0$.

(b)

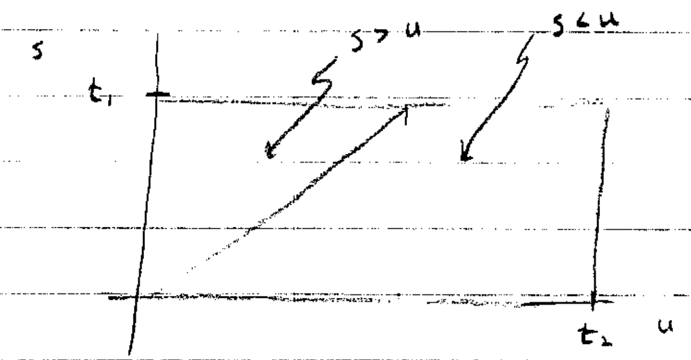
$$R_Y(t_1, t_2) = E[Y(t_1)Y(t_2)]$$

$$= E\left[\int_0^{t_1} X(s)\,ds \int_0^{t_2} X(u)\,du\right]$$

$$= \int_0^{t_1}\int_0^{t_2} E[X(s)X(u)]\,du$$

$$= \int_0^{t_1}\int_0^{t_2} e^{-3|s-u|}\,ds\,du$$

Assume $t_1 < t_2$




$$= 2\int_0^{t_1}\int_0^{s} e^{-3(s-u)}\,du\,ds + \int_{t_1}^{t_2}\int_0^{t_1} e^{-3(u-s)}\,ds\,du$$

$$= 2\int_0^{t_1} e^{-3s}\,\frac{1}{3}e^{3u}\Big|_0^{s}\,ds + \int_{t_1}^{t_2} e^{-3u}\,\frac{1}{3}e^{3s}\Big|_0^{t_1}\,du$$

$$= 2/3\int_0^{t_1}(1 - e^{-3s})\,ds + \frac{1}{3}\int_{t_1}^{t_2}(e^{3t_1}e^{-3u} - e^{-3u})\,du$$

$$= \tfrac{2}{3} t_1 - \tfrac{2}{3}\left(-\tfrac{1}{3} e^{-3s}\right)\Big|_0^{t_1} + \tfrac{1}{3}\left(e^{3t_1}\left(-\tfrac{1}{3} e^{-3u}\right)\right)\Big|_{t_1}^{t_2} + \tfrac{1}{3} e^{-3u}\Big|_{t_1}^{t_2}$$

$$= \tfrac{2}{3} t_1 + \tfrac{2}{9} e^{-3t_1} - \tfrac{2}{9} - \tfrac{1}{9} e^{3t_1 - 3t_2} - \tfrac{1}{9} + \tfrac{1}{3} e^{-3t_2} - \tfrac{1}{3} e^{-3t_1}$$

$$= \tfrac{2}{3} t_1 - \tfrac{1}{9} e^{-3t_1} - \tfrac{1}{3} + \tfrac{1}{3} e^{-3t_2} - \tfrac{1}{9} e^{3(t_1 - t_2)}$$

and a similar result holds for $t_2 < t_1$ (with $t_1$ & $t_2$ reversed) To show it is not wss, let $t_1 = t_2$:

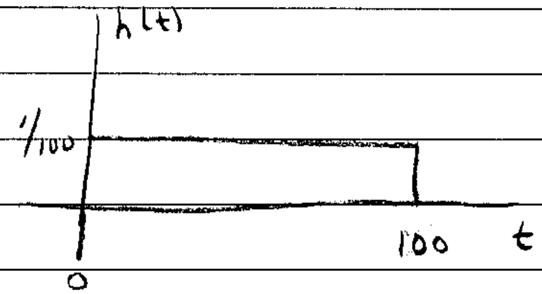
$$R_y(t_1, t_1) = \tfrac{2}{3} t_1 + \tfrac{2}{9} e^{-3t_1} - \tfrac{2}{9}$$

which grows in $t_1 \Rightarrow$ not wss!

(c)

$$S_y(f) = |H(f)|^2 S_x(f)$$

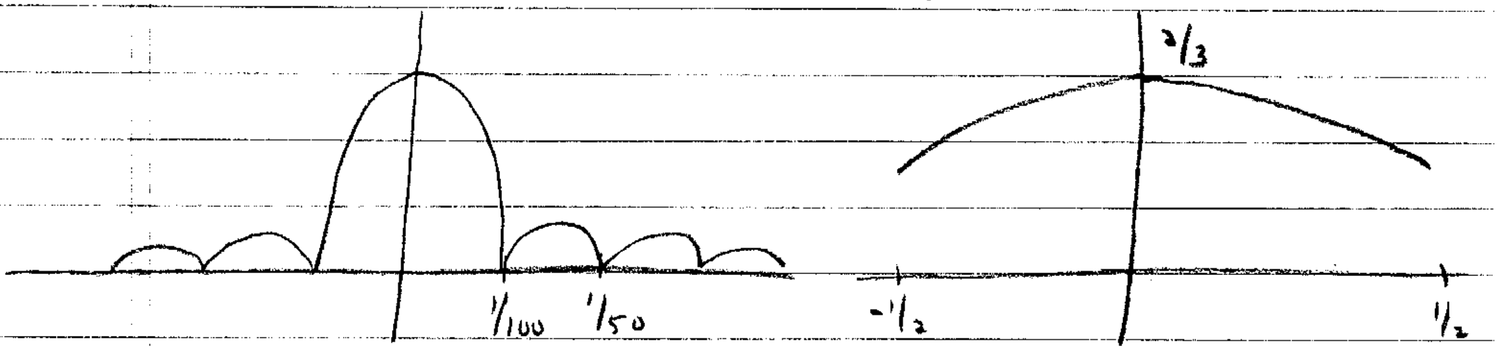
where




$$= \tfrac{1}{100} P\left(\tfrac{(t-50)}{100}\right) \overset{\mathcal{F}}{\Longleftrightarrow} \operatorname{sinc}(100f)\, e^{-j2\pi 50 f}$$

$\Rightarrow$

$$S_Y(f) = \operatorname{sinc}^2(100f)\,\frac{6}{9+4\pi^2 f^2}$$




Thus, $X(t)$ looks almost "white" to this filter, with power spectral density $S_X(f) = 2/3$.

$$P_Y \approx 2/3 \int_{-\infty}^{\infty} \operatorname{sinc}^2(100f)\, df$$

$$= 2/3 \int_{-\infty}^{\infty} 1/10^4 \; p(f/100)\, df$$

$$= 2/3 \cdot 1/100$$

$$= 1/150$$

6)

$$\hat{X}_{ML}(Y=y) = \underset{x}{\text{argmax}}\ P(Y=y \mid \lambda = x)$$

$$= \underset{x}{\text{argmax}}\ \frac{e^{-5x}(5x)^{10}}{10!}$$

$$= \underset{x}{\text{argmax}}\ e^{-5x}(5x)^{10}$$

$$= \underset{x}{\text{argmax}}\ (-5x + 10 \ln 5x)$$

$$d/dx\,(-5x + 10\ln(5x)) = -5 + \frac{10}{5x} \cdot 5 = 0$$

$$x = 2$$

$$\hat{X}_{ML}(Y=10) = 2$$