1. Consider a binary communication system over a 1-dimensional vector channel where message \( m_1 \) is sent by signaling with \( s_1 \), \( m_2 \) is sent by signaling with \( s_2 \) (assume \( s_2 > s_1 \)), \( r = s + n \), and \( n \) is a zero-mean Gaussian random variable with variance \( \frac{N_0}{2} \). Suppose that message \( m_1 \) (and thus \( s_1 \)) is sent with probability \( p \).

(a) Sketch \( p(r|s_1)p(s_1) \) and \( p(r|s_2)p(s_2) \) versus \( r \) on the same plot for \( p = 0.75 \).

(b) For general \( p \):
   - Find the optimal threshold \( c \) that splits the decision regions and state the decision rule.
   - Explain the decision rule for \( p = 0.5 \) (equally likely signals).
   - Explain the decision rule for \( p = 0 \).

(c) For general \( p \), find \( P(E) \), the probability of error of a system employing the decision rule found in (b).

(d) Consider \( p = 0.5 \) and an average transmit energy constraint \( E[|s_1|^2] = E_s \). Find the values of \( s_1 \) and \( s_2 \) that minimize \( P(E) \).

2. Consider the two-dimensional vector channel \( r = s + n \) with the four signal vectors:

\[
\begin{align*}
\mathbf{s}_0 &= \left( \sqrt{\frac{E_s}{2}}, \sqrt{\frac{E_s}{2}} \right)^T \quad \text{Info bits: 00} \\
\mathbf{s}_1 &= \left( -\sqrt{\frac{E_s}{2}}, \sqrt{\frac{E_s}{2}} \right)^T \quad \text{Info bits: 01} \\
\mathbf{s}_2 &= \left( \sqrt{\frac{E_s}{2}}, -\sqrt{\frac{E_s}{2}} \right)^T \quad \text{Info bits: 10} \\
\mathbf{s}_3 &= \left( -\sqrt{\frac{E_s}{2}}, -\sqrt{\frac{E_s}{2}} \right)^T \quad \text{Info bits: 11}
\end{align*}
\]

that correspond to messages that are transmitted with equal probability. The noise vector has independent zero-mean Gaussian components with variance \( \frac{N_0}{2} \).

(a) Using symmetry arguments, find the exact probability of symbol error of this system as a function of \( E_s \) and \( N_0 \).
(b) For large signal-to-noise ratios, the following approximation is valid:

\[ P(E|\xi_{\text{sent}}) \approx N_{\text{min},i} Q \left( \frac{d_{\text{min},i}}{\sqrt{2}N_0} \right) \]

where \( d_{\text{min},i} = \min_{j \neq i} |s_j - s_i| \) and \( N_{\text{min},i} \) is the number of signals such that \( |s_j - s_i| = d_{\text{min}} \).

- Find this approximation for the signal set above.
- Show how the same approximation can be obtained from your solution to (a).

3. Consider the following on-off keying system for transmitting a bit \( b_0 \), equally likely to be 0 or 1, in \( t \in (0, 1) \). For \( b_0 = 0 \), we let \( s(t) = 0 \), and for \( b_0 = 1 \), we let \( s(t) = \sqrt{2E_s} p(t) \), where

\[ p(t) = \begin{cases} 
1 & 0 \leq t \leq 1 \\
0 & \text{otherwise}
\end{cases} \]

The signal \( s(t) \) is transmitted across a channel modeled as the following:

\[ s(t) \rightarrow \text{+} \rightarrow r(t) \]

\[ \downarrow n(t) \]

where \( n(t) \) is additive white Gaussian noise with power spectral density \( \frac{N_0}{2} \) and \( r(t) \) is the received waveform.

(a) Find the receiver for processing \( r(t), t \in (0, 1) \) to obtain an estimate for the transmitted bit that minimizes the probability of a bit error.

(b) Find the probability of a bit error in terms of the average energy per symbol \( E_s \) and \( N_0 \).

(c) How much better or worse (in dB of \( \frac{E_s}{N_0} \)) is this system than a binary phase-shift keyed (BPSK) system operating on an AWGN channel?
4. Consider the waveform channel:

\[
\begin{align*}
    s(t) &\rightarrow + \\
    n(t) &\rightarrow r(t)
\end{align*}
\]

where \(n(t)\) is additive white Gaussian noise with power spectral density \(\frac{N_0}{2}\), \(r(t)\) is the received waveform, and \(s(t) = s_i(t)\) when message \(m_i\) is to be sent during time \(t \in (0, T_s)\). Suppose there are \(M = 4\) possible **equally likely** messages and the corresponding signals are:

\[
\begin{align*}
    s_1(t) &= \sqrt{\frac{2}{T_s}} T_s \\
    s_2(t) &= -\sqrt{\frac{2}{T_s}} T_s \\
    s_3(t) &= 3\sqrt{\frac{2}{T_s}} T_s \\
    s_4(t) &= -\sqrt{\frac{2}{T_s}} T_s
\end{align*}
\]

(a) Find an orthonormal basis \(\{\phi_j(t) : j = 1, \ldots, N\}\) with minimum \(N\) for these signals, and give the vector representation of each of the signals in this basis. (You don’t have to explicitly use Gram-Schmidt, but then you must comment on how you know your basis minimizes \(N\).)

(b) Specify the MAP receiver by:

- Drawing a (simple) block diagram showing a method of obtaining \(r_j\) (the component of \(r(t)\) along the basis function \(\phi_j(t)\)) from \(r(t)\).
- Drawing the decision regions in \(\mathbb{r}\)-space \((\mathbb{r} = (r_1, \ldots, r_N)^T)\), showing where each signal is chosen. You do not have to give precise intercepts and such for the boundaries - just approximate the picture.
(c) Give the Union bound on the symbol error probability of the MAP receiver. It is sufficient to express the bound as a function of the units of the vector space - you need not convert to average energy $E_s$.

(d) Recall that the Union Bound for $P(E)$ is obtained by union bounding $P(E|s_i)$ for each $i$. Consider $P(E|s_3)$. Argue that one of the $M - 1$ terms from the union bound on $P(E|s_3)$ can be removed and still yield an upper bound.

5. Consider the waveform channel:

\[ s(t) + n(t) \rightarrow r(t) \]

where $n(t)$ is additive white Gaussian noise with power spectral density $\frac{N_0}{2}$, $r(t)$ is the received waveform, and $s(t) = s_i(t)$ when message $m_i$ is to be sent during time $t \in (0, T_s)$. Suppose there are $M = 2$ possible equally likely messages and the corresponding signals are:

\[
\begin{align*}
s_1(t) &= \sqrt{2P_c} \cos(2\pi f_c t), \quad 0 \leq t \leq T_s \\
s_2(t) &= -\sqrt{2P_c} \cos(2\pi f_c t), \quad 0 \leq t \leq T_s
\end{align*}
\]

where $P_c$ is the transmitted power and $f_c$ is the carrier frequency.

(a) Specify the optimal (MAP) processing of $r(t), t \in [0, T_s]$ for determining which message was sent. What is the probability of error of the MAP receiver in terms of $E_s = P_c T_s$ and $N_0$?

(b) The same exact processing from (a) is employed but unknown to the receiver, the transmitted signals are really given by

\[
\begin{align*}
s_1(t) &= \sqrt{2P_c} \cos(2\pi f_c t + \theta_e), \quad 0 \leq t \leq T_s \\
s_2(t) &= -\sqrt{2P_c} \cos(2\pi f_c t + \theta_e), \quad 0 \leq t \leq T_s
\end{align*}
\]

where $0 \leq \theta_e \leq \frac{\pi}{2}$. Find the probability of error of this system as a function of $\theta_e$. (Assume $f_c \gg \frac{1}{T_s}$).
6. Consider the following channel:

\[ r_1 = s + n_1 \]
\[ r_2 = n_1 + n_2 \]

where \( r_1 \) and \( r_2 \) are observable at the receiver, and \( n_1 \) and \( n_2 \) are independent zero-mean Gaussian random variables with variance \( \frac{N_0}{2} \). The channel is used to transmit one of two equally likely messages \( m_1 \) and \( m_2 \), where \( s = s_1 = \sqrt{E_s} \) if \( m_1 \) is to be sent, and \( s = s_2 = -\sqrt{E_s} \) if \( m_2 \) is to be sent.

(a) Find the MAP decision rule. Explain why it depends on \( r_2 \), even though \( r_2 \) "is only noise."

(b) Find the probability of error of the MAP receiver.

(c) Suppose we discarded signal \( r_2 \) because "it is only noise." How much more energy (in dB) would we have to use with this suboptimal system as compared to the system of (a) to achieve the same probability of error?