1. You have a table that gives you the value of the “Goeckel Function” for all \( x \geq 0 \):

\[
G(x) = \int_x^\infty \frac{1}{2} e^{-\frac{u^2}{2}} du
\]

Let \( Y \) be a Gaussian random variable with mean \( \mu \) and variance \( \sigma^2 \); that is, \( Y \sim N(\mu, \sigma^2) \). Write an expression for \( P(Y \leq y) \) for all \( y \) in terms of \( G(x) \).

2. Let \( X \) and \( Y \) be jointly Gaussian random variables with \( E[X] = 0 \), \( E[Y] = 0 \), \( \text{Var}[X] = 5 \), \( \text{Var}[Y] = 10 \), and \( E[XY] = 2 \). Let \( Z = 2X + 3Y \). Find the \( f_Z(z) \), the probability density function of \( Z \).

3. A main utility of wide-sense stationarity (WSS) is that it allows us to define the power spectral density for a given random process. However, there is a large class of random processes that are not wide-sense stationary for which we are still able to derive the power spectral density.

A random process \( Y(t) \) is defined to be cyclostationary if both its mean function \( m_Y(t) \) and the autocorrelation function \( R_Y(t, t + \tau) = E[Y(t)Y(t + \tau)] \) are periodic (in \( t \)) at some period \( T_0 \); that is, \( m_Y(t) = m_Y(t + T_0) \) and \( R_Y(t, t + \tau) = R_Y(t + T_0, t + T_0 + \tau) \) for some \( T_0 \).

The “average autocorrelation” of a cyclostationary process \( Y(t) \) is given by:

\[
\bar{R}_Y(\tau) = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} R_Y(t, t + \tau) dt
\]

and the power spectral density \( S_Y(f) \) of a cyclostationary process \( Y(t) \) is given by the Fourier transform of \( \bar{R}_Y(\tau) \).

Let \( M(t) \) be a zero-mean wide-sense stationary random process with autocorrelation function \( R_M(\tau) \), and let

\[
X(t) = M(t) \cos(2 \pi f_c t)
\]

where \( f_c \) is some constant. Show \( X(t) \) is cyclostationary and use the definitions above to derive the power spectral density for \( X(t) \).

4. Let \( X(t) \) be a wide-sense stationary Gaussian random process with mean zero and autocorrelation \( R_X(\tau) = e^{-|\tau|/\tau} \). Let \( N(t) \) be a white Gaussian noise process with power spectral density \( \frac{N_0}{2} \).

(a) Find \( P_x \), the power in \( X(t) \).

(b) Is \( X(t) \) strict-sense stationary?

(c) Find \( P(X(3) > 2) \).
(d) Find the power spectral density $S_X(f)$ of $X(t)$.

(e) Find a filter (give $h(t)$ or $H(f)$) that has input $N(t)$ and output with power spectral density $S_X(f)$.

(f) Let $Z = X(0) + X(1) + X(2)$. Find $f_Z(z)$, the pdf of $Z$.

(g) Find $P(X(0) + X(1) > 3)$.

5. The letters $z_0, z_1, z_2, z_3, z_4, z_5, z_6$ come into your lossless source coder with probabilities 0.30, 0.25, 0.15, 0.1, 0.1, 0.05, 0.05, respectively.

(a) Find a Huffman code when you take $N = 1$ input symbol per block.

(b) Find the rate (in bits/symbol) of your Huffman coder of part (a).

6. [ECE 645 only] A Huffman code finds the optimal codeword to assign to a given block of $N$ source symbols.

(a) Show that \{01, 100, 101, 1110, 1111, 0011, 0001\} cannot be a Huffman code for any $N$ for any source distribution where every string to be coded has non-zero probability.

(b) For a source producing an IID sequence of discrete random variables, each drawn from source alphabet $\mathcal{X}$, it has been found that a Huffman code on blocks of length 2 (i.e. $N = 2$ source symbols are taken at a time) has rate 2 bits/symbol, and a Huffman code on blocks of length 3 (i.e. $N = 3$ source symbols are taken at a time) has rate 1.6 bits/symbol.

- Find upper and lower bounds to the first-order entropy of the source.
- What can be deduced about the size of the source alphabet?

(c) Does there exist a source producing an IID sequence of discrete random variables, and integers $N \in \{1, 2, 3, 4, \ldots\}$ and $M \in \{2, 3, 4, \ldots\}$, such that a Huffman code on blocks of length $N$ has (strictly) smaller rate (in bits/symbol) than a Huffman code on blocks of length $MN$?

7. Consider the two functions $s_1(t)$ and $s_2(t)$ given by:

- $s_1(t) = t$ for $0 \leq t \leq 1$ and zero elsewhere
- $s_2(t) = t^2$ for $0 \leq t \leq 1$ and zero elsewhere

(a) Construct an orthonormal basis for these functions using the Gram-Schmidt procedure starting with $s_1(t)$.

(b) In this basis, give the vector representations of $s_1(t)$ and $s_2(t)$.

(c) Compute
\[ \int_{-\infty}^{\infty} (s_1(t) - s_2(t))^2 dt \]

using the vector representation. Compare your result with the value obtained by direct integration.

(d) Show, for general \( s_i(t) \) and \( s_j(t) \), that the waveform distance squared is equal to the distance squared between the vectors that give the signals’ representation on an orthonormal basis.