Overview

• The exam consists of four problems for 100 points (ECE 564) or 120 points (ECE 645). The points for each part of each problem are given in brackets - you should spend your two hours accordingly.

• The exam is closed book, but you are allowed three page-sides of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

• Full credit will be given only to fully justified answers.

• Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.

• If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.

• Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.

• If you get to the end of the problem and realize that your answer must be wrong, be sure to write “this must be wrong because . . .” so that I will know you recognized such a fact.

• Academic dishonesty will be dealt with harshly - the minimum penalty will be an “F” for the course.
Some (Possibly) Useful Identities:

\[
\begin{align*}
\cos(A + B) &= \cos(A) \cos(B) - \sin(A) \sin(B) \\
\cos(A - B) &= \cos(A) \cos(B) + \sin(A) \sin(B) \\
\sin(A + B) &= \sin(A) \cos(B) + \cos(A) \sin(B) \\
\sin(A - B) &= \sin(A) \cos(B) - \cos(A) \sin(B) \\
\sin^2(A) &= \frac{1}{2} - \frac{1}{2} \cos(2A) \\
\cos^2(A) &= \frac{1}{2} + \frac{1}{2} \cos(2A) \\
\cos(A) \cos(B) &= \frac{1}{2} \cos(A + B) + \frac{1}{2} \cos(A - B) \\
\sin(A) \sin(B) &= \frac{1}{2} \cos(A - B) - \frac{1}{2} \cos(A + B) \\
\sin(A) \cos(B) &= \frac{1}{2} \sin(A + B) + \frac{1}{2} \sin(A - B)
\end{align*}
\]
1. Consider an independent and identically distributed (IID) discrete source \( \{ \hat{X}_k \} \) that can assume one of \( M = 6 \) values, where each source symbol has the following probability assignment:

\[
p_{\hat{X}_k}(x) = \begin{cases} 
0.4, & x = A \\
0.2, & x = B \\
0.2, & x = C \\
0.1, & x = D \\
0.06, & x = E \\
0.04, & x = F \\
0 & \text{otherwise}
\end{cases}
\]

[10] (a) Suppose that the source symbols are losslessly encoded one at a time \( (N = 1) \). Find two (different) Huffman codes with distinct length distributions. In other words, the set \( \{ n(B_i) : i = 1, \ldots, M \} \) for one of your Huffman codes must be different than it is for the other code, where \( n(B_i) \) is the length of the codeword assigned to the block \( B_i \).

[3] (b) Find the rate of each of your Huffman codes from (a).

[7] (c) Suppose a friend designs a Huffman code for this source while taking three blocks \( (N = 3) \) at a time. Using the rate of your Huffman codes from (a), what lower and upper bounds can you give on the rate of such a \( N = 3 \) Huffman code?

2. [10] (a) Suppose that we have a signal set \( \{ s_i(t) : i = 1, \ldots, M \} \) defined for \( t \in [0,T_s] \). We perform Gram-Schmidt orthogonalization to find an orthonormal basis \( \{ \phi_j(t) : j = 1, \ldots, N \} \), \( N \leq M \), for the signal set. Gram-Schmidt also provides the vectors \( \{ s_i : i = 1, \ldots, M \} \) such that

\[
s_i(t) = \sum_{j=1}^{N} s_{i,j} \phi_j(t)
\]

Show that

\[
\int_0^{T_s} (s_m(t) - s_n(t))^2 dt = \sum_{j=1}^{N} (s_{m,j} - s_{n,j})^2
\]

for any \( m \) and \( n \).

[10] (b) (Be sure to see the big hint at the end before starting!). Consider the waveform channel:

\[
s(t) \rightarrow + \rightarrow r(t) \rightarrow n(t)
\]
where \( n(t) \) is additive white Gaussian noise with power spectral density \( \frac{N_0}{2} \), \( r(t) \) is the received waveform, and \( s(t) = s_i(t) \) when message \( m_i \) is to be sent during time \( t \in [0, T_s] \). Suppose there are \( M = 3 \) possible equally likely messages and the corresponding signals are:

\[
\begin{align*}
    s_1(t) &= \sqrt{\frac{2E_s}{T_s}} \cos (2\pi f_c t + \theta_0), \quad 0 \leq t \leq T_s \\
    s_2(t) &= \sqrt{\frac{2E_s}{T_s}} \cos (2\pi (f_c + \Delta f) t + \theta_0), \quad 0 \leq t \leq T_s \\
    s_3(t) &= \sqrt{\frac{2E_s}{T_s}} \cos (2\pi (f_c + 2\Delta f) t + \theta_0), \quad 0 \leq t \leq T_s
\end{align*}
\]

where \( T_s, E_s, f_c, \Delta f, \) and \( \theta_0 \) are all known at the receiver.

Find the Union Bound to the probability of symbol error of the optimal (MAP) receiver in terms of \( \Delta f T_s \) and \( \frac{E_s}{N_0} \). (Assume \( f_c \gg \Delta f \) and \( f_c \gg \frac{1}{T_s} \).) (Big hint: You do not need to find an orthonormal basis. Rather, use your result from (a) to recognize that you can find the distances you need for your Union bound formula directly from the waveforms.)

3. Some Error Control Coding Questions:

[5] (a) We know that a linear block code must contain the all 0’s codeword (a zero weight codeword). Is it possible to have a linear block code for which all non-zero codewords have odd weight?

[5] (b) I am seeking a (7,2) double-error correcting (DEC) code. I check the Hamming bound and see that

\[
\binom{7}{2} + \binom{7}{1} + 1 \leq 32 = 2^{(n-k)}
\]

Can I then say that such a code certainly exists? If you say “yes”, tell me what the left and right sides mean in the equation above in terms of code parameters. If you say “no”, be sure to tell me why the answer is “no” yet the Hamming bound is still satisfied.

[5] (c) What is the minimum distance of a (16,11) code obtained by deleting all of the even weight columns of the parity check matrix of the (31,26) Hamming code? Recall that the parity check matrix of the (31,26) Hamming code has 31 columns that consist of all of the non-zero binary 5-tuples.

4. Suppose we have a (5,2) linear block code with the following generator matrix:

\[
G = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

(a) Code basics:

- [2] What is the rate \( r \) of this code?
• [2] How many syndromes are there?
• [2] How many errors can this code correct? (Recall that, for a code to be $t$-error correcting, it must correct every pattern of $t$ or fewer errors.)

[5] (b) List the syndromes and the associated coset leaders.

The output from our channel coder goes to a modulator, which sends each coded bit with one use of a 2-dimensional vector additive white Gaussian noise (AWGN) channel:

$$r = s + n$$

where $r$ is the received column vector, and the noise is given by $n = (n_1, n_2)^T$. The noise components $n_1$ and $n_2$ are independent Gaussian random variables with mean 0 and variance $\frac{N_0}{2}$. In other words, this is a two-dimensional AWGN vector channel, and each coded bit is sent with a separate use of this channel.

If the coded bit is a 0, $s_0 = (1, -2)^T$ is sent. If the coded bit is a 1, $s_1 = (2, -1)^T$ is sent.

Hence, when two information bits go into the encoder, five bits come out of the encoder that are then mapped to ten (real) channel symbols.

[7] (c) If hard-decision decoding is done at the decoder (i.e. the demodulator processes the two-dimensional vector $r$ corresponding to each coded bit to choose the most likely transmitted bit and passes only this bit decision to the decoder, which uses five bit decisions to choose the transmitted codeword), find the exact codeword error probability $P(E)$. Leave your answer in terms of the units of the signal space and $N_0$ (i.e. there is no need to convert to $E_s$ or $E_b$).

[8] (d) If optimal soft-decision decoding is done at the decoder (i.e. the decoder makes the optimal choice of a codeword based on the sequence of five received two-dimensional vectors $r$, or ten dimensions total), find a Union Bound to the codeword error probability given the codeword 00000 was transmitted. Leave your answer in terms of the units of the signal space and $N_0$ (i.e. there is no need to convert to $E_s$ or $E_b$).

[5] (e) Convert your answers to (c) and (d) to be in terms of the energy per information bit $E_b$ and $N_0$.

[10] (f) This is a pretty lousy modulation scheme in two ways: (i) codeword error probability versus $E_b/N_0$ is poor, and (ii) low rate (only two information bits in 10 total channel dimensions!). Keeping the code the same, specify a better modulation scheme that takes at most 5 total channel dimensions (something simple from class is fine). How much better is your scheme in dB than the scheme given above for soft-decision decoding at high signal-to-noise ratios (SNRs)? (Most of the points for this part of the problem are for the characterization of your scheme’s performance.)
5. [645 only] Consider the 1-dimensional vector channel \( r = s + n \) where \( r \) is the received vector, \( s = s_1 = -\sqrt{E_s} \) if \( m_1 \) is sent and \( s = s_2 = \sqrt{E_s} \) if \( m_2 \) is sent, and \( n \) is a Laplacian random variable with probability density function

\[
p_n(x) = \frac{1}{2\lambda} e^{-|x|/\lambda}
\]

Message \( m_1 \) is sent with probability \( p \) and message \( m_2 \) is sent with probability \( 1 - p \).

[12] (a) We know that the MAP decision rule finds the message \( m_i \) such that \( P(s_i|r) \) is maximized. Find the MAP decision rule as a function of \( p \) and draw the decision regions in \( r \)-space where each signal is chosen. (If you cannot get this for general \( p \), do be sure to at least note the regions for \( p = 1/2 \) and then do part (b).)

[8] (b) Find the symbol error probability \( P(E) \) for \( p = 1/2 \) as a function of \( E_s \) and \( \lambda \).