Midterm 1 Solutions

ECE 314 — Spring 2012

1)

$Q_1, \bar{Q}_1$ a partition

(a) $P(Q_2) \overset{\text{Law of Total Probability}}{=} P(Q_2|Q_1)P(Q_1) + P(Q_2|\bar{Q}_1)P(\bar{Q}_1)$

$= 0.9 \cdot 0.9 + 0.3 \cdot 0.1 = 0.84$

(b) $P(Q_1|Q_2) \overset{\text{Bayes Rule}}{=} \dfrac{P(Q_2|Q_1)P(Q_1)}{P(Q_2)} = \dfrac{0.9 \cdot 0.9}{0.84} = \dfrac{0.81}{0.84}$

(c) $P(Q_1|Q_3) \overset{\text{independent}}{=} P(Q_1) = 0.9$

(d) • Need Q_1 <u>and</u> (Q_2 <u>or</u> Q_3)

$A = Q_1 \cap (Q_2 \cup Q_3)$

• $P(A) = ((Q_1 \cap Q_2) \cup (Q_1 \cap Q_3))$

$= P(Q_1 \cap Q_2) + P(Q_1 \cap Q_3) - P(\overbrace{(Q_1 \cap Q_2) \cap (Q_1 \cap Q_3)}^{Q_1 \cap Q_2 \cap Q_3})$

$= P(Q_2|Q_1)P(Q_1) + P(Q_1)P(Q_3) - P(Q_3)P(Q_2|Q_1)P(Q_1)$

$= 0.9 \cdot 0.9 + 0.9 \cdot 0.5 - 0.5 \cdot 0.9 \cdot 0.9$

$= 0.81 + 0.45 - 0.405$

$= 0.855$

(e) $X \in \{0, 10, 20, 30, 40\}$

$P(X=0) = P(\bar{Q}_1 \cap \bar{Q}_2 \cap \bar{Q}_3) = P(\bar{Q}_3)P(\bar{Q}_2|\bar{Q}_1)P(\bar{Q}_1) = 0.5 \cdot 0.7 \cdot 0.1$
$= 0.035$

$P(X=10) = P((\bar{Q}_1 \cap Q_2 \cap \bar{Q}_3) \cup (\bar{Q}_1 \cap \bar{Q}_2 \cap Q_3))$

$\overset{\text{disjoint}}{=} P(\bar{Q}_3)P(Q_2|\bar{Q}_1)P(\bar{Q}_1) + P(Q_3)P(\bar{Q}_2|\bar{Q}_1)P(\bar{Q}_1)$

$= 0.5 \cdot 0.3 \cdot 0.1 + 0.5 \cdot 0.7 \cdot 0.1 = 0.05$

$$P(X=20) = P((\overline{Q}_1 \cap Q_2 \cap Q_3) \cup (Q_1 \cap \overline{Q}_2 \cap \overline{Q}_3))$$

$$\overset{\text{disjoint}}{=} P(Q_3)P(Q_2|\overline{Q}_1)P(\overline{Q}_1) + P(\overline{Q}_3)P(\overline{Q}_2|Q_1)P(Q_1)$$

$$= 0.5 \cdot 0.3 \cdot 0.1 + 0.5 \cdot 0.1 \cdot 0.9$$

$$= 0.06$$

$$P(X=30) = P((Q_1 \cap \overline{Q}_2 \cap Q_3) \cup (Q_1 \cap Q_2 \cap \overline{Q}_3))$$

$$= P(Q_3)P(\overline{Q}_2|Q_1)P(Q_1) + P(\overline{Q}_3)P(Q_2|Q_1)P(Q_1)$$

$$= 0.5 \cdot 0.1 \cdot 0.9 + 0.5 \cdot 0.9 \cdot 0.9$$

$$= 0.45$$

$$P(X=40) = P(Q_1 \cap Q_2 \cap Q_3)$$

$$= P(Q_3)P(Q_2|Q_1)P(Q_1)$$

$$= 0.5 \cdot 0.9 \cdot 0.9$$

$$= 0.405$$

2) (a) Lots of ways to do this - some counting and some not:

Let N_i: no repeat on i^{th} roll

A: no repeats

$$P(A) = P(N_1 \cap N_2 \cap N_3)$$

$$= P(N_3 | N_2, N_1)\, P(N_2 | N_1)\, P(N_1)$$

$$= 1/3 \cdot 2/3 \cdot 1 = 2/9$$

(Also, could do

$$\frac{\#\text{ ordered ways - no repeats}}{\#\text{ ordered ways}} = \frac{3!}{3^3} = \frac{6}{27} = 2/9 \Big)$$

(b) $P(R) = 1/3 \quad P(\bar{R}) = 2/3$

Bernoulli Trials

$$\binom{20}{8}\left(1/3\right)^8 \left(2/3\right)^{12}$$

(c) A: at least one of each color

R: I see at least one red

G: " " " " " green

B: " " " " " blue

$$P(A) = P(R \cap G \cap B) = 1 - P(\bar{R} \cup \bar{G} \cup \bar{B})$$

$$= 1 - \left(P(\bar{R} \cup \bar{G}) + P(\bar{B}) - P((\bar{R} \cup \bar{G}) \cap \bar{B})\right)$$

$$= 1 - \left(P(\bar{R}) + P(\bar{G}) - P(\bar{R} \cap \bar{G}) + P(\bar{B}) - P((\bar{R} \cap \bar{B}) \cup (\bar{G} \cap \bar{B}))\right)$$

$$= 1 - \Big(P(\bar{R}) + P(\bar{G}) - P(\bar{R} \cap \bar{G})$$
$$- P(\bar{R} \cap \bar{B}) - P(\bar{G} \cap \bar{B}) + P\big(\cancelto{0}{(\bar{R} \cap \bar{B}) \cap (\bar{G} \cap \bar{B})}\big)\Big)$$

$$= 1 - P(\bar{R}) - P(\bar{G}) - P(\bar{B})$$
$$+ P(\bar{R} \cap \bar{G}) + P(\bar{R} \cap \bar{B}) + P(\bar{G} \cap \bar{B})$$

$$= 1 - (2/3)^n - (2/3)^n - (2/3)^n$$
$$+ (1/3)^n + (1/3)^n + (1/3)^n$$

$$= 1 - 3 \cdot (2/3)^n + 3 \cdot (1/3)^n$$

(<u>Note</u>: $P(A) = 2/9$ for $n = 3$).

3)

(a) A: event ≥ 9 parts work G: good machine

$$P(A) = P(A|G)P(G) + P(A|B)P(B)$$

$$= \left(0.9^{10} + \binom{10}{1} 0.9^{9} 0.1^{1}\right) \cdot 0.8$$

$$+ \left(0.3^{10} + \binom{10}{1} 0.3^{9} 0.7^{1}\right) \cdot 0.2$$

(b) $$P(G|A) \underset{\text{Bayes}}{=} \frac{P(A|G)P(G)}{P(A)} = \frac{\left(0.9^{10} + \binom{10}{1} 0.9^{9} 0.1^{1}\right) \cdot 0.8}{\underbrace{P(A)}_{\text{from part (a)}}}$$

(c) C: decision is correct

$$P(C) = P(C|G)P(G) + P(C|B)P(B)$$

$$= P(\text{"1st part is good"} | G)P(G)$$

$$+ P(\text{"1st part is bad"} | B)P(B)$$

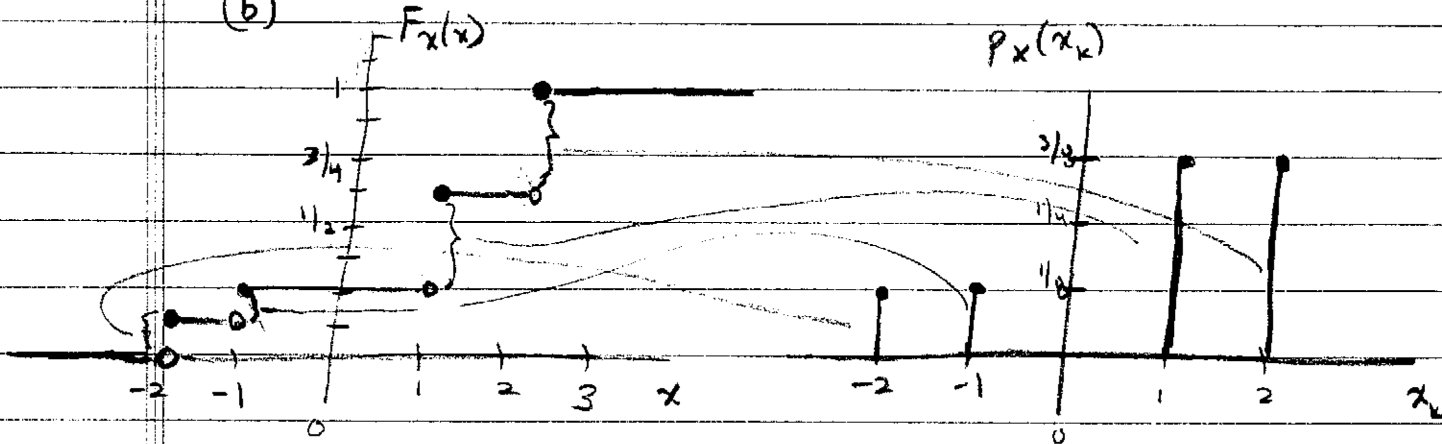
$$= 0.9 \cdot 0.8 + 0.7 \cdot 0.2 = 0.86$$

4)

(a) cdfs always go to 1 as $x \to \infty$

$$\Rightarrow 8c = 1 \Rightarrow c = 1/8$$

(b)



Jumps in CDF are "sticks" in pmf

(c) $P(X \le 1) = p_X(-2) + p_X(-1) + p_X(1)$

all "sticks" above that interval

$$= 1/8 + 1/8 + 3/8$$

$$= 5/8$$

(d) $$P(\{X \ge -1\} \mid \{X \le 1\}) = \frac{P(\{X \ge -1\} \cap \{X \le 1\})}{P(\{X \le 1\})} = \frac{p_X(-1) + p_X(1)}{5/8}$$

$$= 4/8 \big/ 5/8 = 4/5$$

(e) Law of Total Prob

$$P(Y > 1.5) = \sum_{x_k} P(Y > 1.5 \mid X = x_k)\, P(X = x_k)$$

$$= \underbrace{1/4 \cdot 1/8}_{x_k = -2} + \underbrace{0 \cdot 1/8}_{x_k = -1} + \underbrace{0 \cdot 3/8}_{x_k = 1} + \underbrace{1/4 \cdot 3/8}_{x_k = 2} = 1/8$$