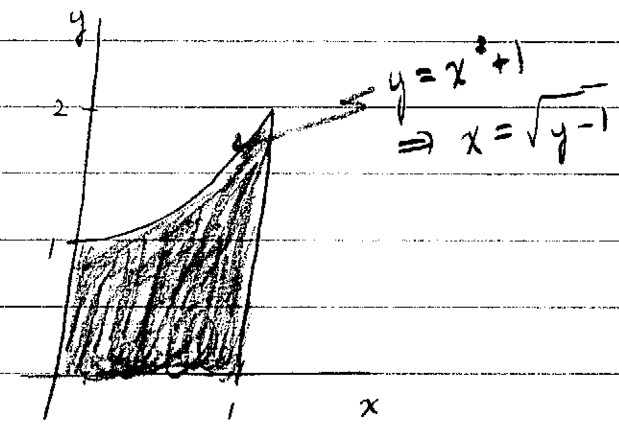
ECE 314 Spring 2012

1)

(a)



(b) $$\int_0^1\int_0^{x^2+1} kx\, dy\, dx$$

$$= \int_0^1 (kxy)\Big|_0^{x^2+1} dx = \int_0^1 kx(x^2+1)\,dx$$

$$= k\int_0^1 (x^3+x)\,dx = k\left(\tfrac{1}{4}x^4 + \tfrac{1}{2}x^2\right)\Big|_0^1 = \tfrac{3}{4}k \Rightarrow k = \tfrac{4}{3}$$

(c) $0 \le x \le 1$

$$f_X(x) = \int_0^{x^2+1} \tfrac{4}{3}x\, dy = \left(\tfrac{4}{3}xy\right)\Big|_0^{x^2+1} = \tfrac{4}{3}x^3 + \tfrac{4}{3}x$$

$$\Rightarrow f_X(x) = \begin{cases} \tfrac{4}{3}x^3 + \tfrac{4}{3}x, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

Check: Integrates to 1 ✓

$0 \le y \le 1$

$$f_Y(y) = \int_0^1 \tfrac{4}{3}x\, dx = \left(\tfrac{2}{3}x^2\right)\Big|_0^1 = \tfrac{2}{3}$$

$1 \le y \le 2$

$$f_Y(y) = \int_{\sqrt{y-1}}^1 \tfrac{4}{3}x\, dx = \left(\tfrac{2}{3}x^2\right)\Big|_{\sqrt{y-1}}^1 = \tfrac{2}{3} - \tfrac{2}{3}(y-1)$$

$$= \tfrac{4}{3} - \tfrac{2}{3}y$$

(continued)

$$\Rightarrow f_Y(y) = \begin{cases} 2/3, & 0 \le y \le 1 \\ 4/3 - 2/3\, y, & 1 \le y \le 2 \\ 0, & \text{else} \end{cases}$$

Check

$$\int_0^1 2/3 \, dy + \int_1^2 (4/3 - 2/3\,y)\, dy = 2/3 + \underbrace{\left(4/3\, y - 1/3\, y^2\right)\Big|_1^2}_{8/3 - 4/3 - 4/3 + 1/3} = 1 \quad \checkmark$$

(d) $$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

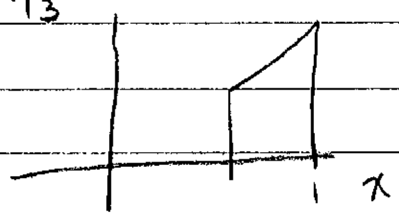
$$= \begin{cases} 4/3\, x \big/ 2/3 = 2x & 0 \le y \le 1, \ 0 \le x \le 1 \\ 4/3\, x \big/ (4/3 - 2/3\, y) = & 1 \le y \le 2 \\ \quad x/(1 - y/2), & \sqrt{y-1} \le x \le 1 \\ 0, & \text{else} \end{cases}$$

Check:

For $0 \le y \le 1$, $\int_0^1 2x \, dx = x^2 \Big|_0^1 = 1 \ \checkmark$

For $1 \le y \le 2$, $\displaystyle\int_{\sqrt{y-1}}^1 \frac{x}{(1-y/2)}\, dx = \frac{1/2\, x^2 \Big|_{\sqrt{y-1}}^1}{(1-y/2)} = \frac{1/2 - \frac{1}{2}(y-1)}{1 - y/2} = 1 \ \checkmark$

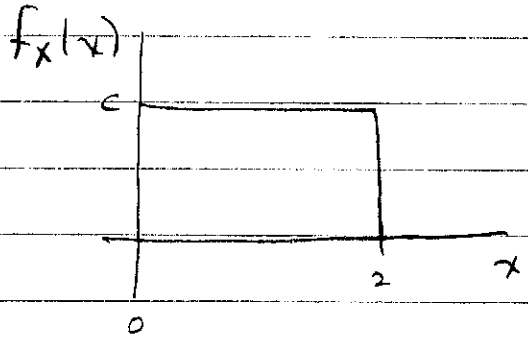
(e) $$f_{X|Y}(x|1.5) = \frac{4/3\, x}{1/3} = 4x, \quad \sqrt{1/2} \le x \le 1$$



$x_{opt} = 1$

2)

(a)



- $c = 1/2$, so that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$
- $f_{X+2}(x)$: just shift right by 2

$$f_{X+2}(x) = \begin{cases} 1/2, & 2 \le x \le 4 \\ 0, & \text{else} \end{cases}$$

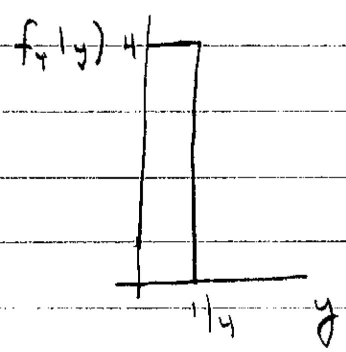
- $f_{-X}(x)$: just flip around $x=0$

$$f_{-X}(x) = \begin{cases} 1/2, & -2 \le x \le 0 \\ 0, & \text{else} \end{cases}$$

- $f_{2X}(x)$: uniform between 0 and 4

$$f_{2X}(x) = \begin{cases} 1/4, & 0 \le x \le 4 \\ 0, & \text{else} \end{cases}$$

(b)



X, for many reasons

For example

$E[X] = 1$

$E[Y] = 1/8$

linearity of expectation

(c) $E[(X-Y)^2] = E[X^2] - 2E[XY] + E[Y^2]$

$\overset{X,Y \text{ independent}}{=} E[X^2] - 2E[X]E[Y] + E[Y^2]$

$E[X^2] = \int_0^2 \frac{1}{2} x^2 \, dx = \frac{1}{6} x^3 \Big|_0^2 = 4/3$

$E[Y^2] = \int_0^{1/4} 4y^2 \, dy = \frac{4}{3} y^3 \Big|_0^{1/4} = 1/48$

$\Rightarrow E[(X-Y)^2] = 4/3 - 2 \cdot 1 \cdot 1/8 + 1/48$

$= 4/3 - 1/4 + 1/48 = 64/48 - 12/48 + 1/48$

$= 53/48$

(d) Z could be $1/4, 3/4, 5/4,$ or $7/4$

$P(Z = 1/4) = P(0 < X < 1/2) = 1/4$

(others are similar)

$$\Rightarrow p_Z(z_j) = \begin{cases} 1/4, & z_j = 1/4, 3/4, 5/4, 7/4 \\ 0, & \text{else} \end{cases}$$

3)

(a) $$P(X=k) = \underbrace{P(X=k \mid C)}\,P(C) + \underbrace{P(X=k \mid B)}\,P(B)$$

Bernoulli

$$= \binom{3}{k} 0.1^k 0.9^{3-k} \cdot 0.5$$

$$+ \binom{3}{k} 0.5^k 0.5^k \cdot 0.5$$

(b) $$P(B \mid E_1, E_2) = \frac{P(E_1, E_2 \mid B)\, P(B)}{P(E_1, E_2)}$$

$$= \frac{0.5^2 \cdot 1/2}{0.1^2 \cdot 1/2 + 0.5^2 \cdot 1/2}$$

(c) $$P(E_3 \mid E_1, E_2) = \frac{P(E_1 \cap E_2 \cap E_3)}{P(E_1 \cap E_2)}$$

$$= \frac{0.1^3 \cdot 1/2 + 0.5^3 \cdot 1/2}{0.1^2 \cdot 1/2 + 0.1^2 \cdot 1/2}$$

4) (a)

$$\bullet\ P\left(\sum_{i=1}^{500} X_i \geq 1800\right)$$

$$= P\left(\sum_{i=1}^{500} X_i - 1750 \geq 1800-1750\right)$$

$$= P\left(\frac{\sum_{i=1}^{500} X_i - 1750}{\sqrt{500\cdot 3}} \geq \frac{1800-1750}{\sqrt{500\cdot 3}}\right)$$

Y is Gaussian, $\mu=0$, $\sigma^2=1$

$$= P\left(Y \geq \frac{50}{\sqrt{500\cdot 3}}\right) = P\left(Y \geq \sqrt{5/3}\right)$$

$$= 1 - P\left(Y \leq \sqrt{5/3}\right)$$

$$= 1 - \Phi\left(\sqrt{5/3}\right)$$

(b) $\hat{\mu} = 900/300 = 3.0$

$$\bullet\ \sigma^2/N = 3/300 = 1/100$$

conf interval $\approx \left[\hat{\mu} - 2\sqrt{\sigma^2/N},\ \hat{\mu} + 2\sqrt{\sigma^2/N}\right]$

$$= [2.8, 3.2]$$

- <u>Unfair</u>! A fair die would have mean 3.5, which is well outside the confidence interval (actually, 5σ from $\hat{\mu}$).