Multivariate Calculus:

A function of two variables e.g., \( g(x,y) = x^3 + y^2 + 3xy \)

(partial) derivative w.r.t. \( x \): \( \frac{\partial}{\partial x} g(x,y) = 3x^2 + 3y \)

\( y \) constant

dervative w.r.t. \( y \) \( \frac{\partial}{\partial y} g(x,y) = 2y + 3x \)

Same idea with integrals:

\[
\int_{a}^{b} x y^2 \, dx = y^2 \int_{a}^{b} x \, dx = y^2 \left[ \frac{1}{2} x^2 \right]_{a}^{b} = y^2 \left( \frac{1}{2} b^2 - \frac{1}{2} a^2 \right)
\]

Double Integrals:

\[
\iint_{D} g(x,y) \, dA = \int_{a}^{b} \int_{c}^{d} g(x,y) \, dx \, dy
\]

Volume of the region

Note: Choosing \( g(x,y) = 1 \) \( \Rightarrow \int_{D} dxdy = \text{area of } D \)

Example 1: Compute the integral of \( g(x,y) = xy^2 \)

over a rectangular area \( 0 \leq x \leq 2 \) \& \( 0 \leq y \leq 1 \)

\[
\iint_{D} g(x,y) \, dxdy = \int_{0}^{2} \int_{0}^{1} xy^2 \, dx \, dy
\]
\[ I = \int_0^2 y^2 \, dx = \frac{1}{2} \int_0^2 x^2 \, dy \bigg|_{y=0}^{y=1} = \frac{1}{2} \int_0^2 (x^2 - 2y^2) \, dx \]

\[ I = \int_0^1 2y^2 \, dy = \frac{2}{3} \int_0^1 y^3 \, dy = \frac{2}{3} \]

We can change the order of integration first w.r.t. \( y \):

\[ I = \int_0^1 xy^2 \, dy = \frac{1}{3} \int_0^1 y^3 \, dy = \frac{1}{3} \int_0^1 x \, dx = \frac{1}{6} \int_0^2 x^2 \, dx = \frac{2}{3} \]

Rectangular regions are easy.

\[ a \leq x \leq b, \quad c \leq y \leq d \rightarrow \text{the limits of } x \text{ and } y \text{ do not depend on each other.} \]

**Example 2**
\[ \iint_D g(x,y) \, dx \]

*Note: Outer limits must be constant:*

\[ \begin{align*}
  \iint_{D} xy^2 \, dy \, dx &= \int_0^{x/2} \int_{x/2}^{y} xy^2 \, dy \, dx \\
  &= \int_0^{x/2} \left[ \frac{1}{3} xy^3 \right]_{x/2}^{y} \, dx \\
  &= \frac{1}{3} x \left( \frac{x^2}{8} \right) - \frac{1}{24} x^4 \\
  &= \frac{1}{24} x^4 \\
  \Rightarrow \int_{x=0}^{2} \frac{x^4}{24} \, dx &= \frac{1}{120} x^5 \bigg|_0^2 = \frac{1}{120} (32) = \frac{1}{15} 
\end{align*} \]

The other way:

\[ \iint_D xy^2 \, dx \, dy = \int_0^2 \int_{2y}^{1} xy^2 \, dx \, dy \]

\[ \begin{align*}
  I &= \int_{2y}^{1} xy^2 \, dx = \frac{1}{2} y^2 x^2 \bigg|_{2y}^{1} = \frac{1}{2} y^2 (4 - 4y^2) \\
  \Rightarrow \int_{y=0}^{1} \left( 4y^2 - 4y^4 \right) \, dy &= \left[ \frac{2}{3} y^3 - \frac{2}{5} y^5 \right]_0^1 = \frac{2}{3} - \frac{2}{5} = \frac{4}{15} 
\end{align*} \]

Example 3:

\[ g(x,y) = x^2 - y \text{ on the shaded region.} \]
\[
\begin{align*}
I &= \int_{-1}^{1} \int_{\frac{1}{2}x^2}^{\frac{1}{2}x^2} (x^2 - y) \, dy \, dx \\
&= \int_{-1}^{1} \left[ \frac{1}{2}x^2 y - \frac{1}{2}y^2 \right]_{\frac{1}{2}x^2}^{\frac{1}{2}x^2} \, dx \\
&= \int_{-1}^{1} \left( \frac{1}{2}x^4 - \frac{1}{2} \cdot \frac{1}{4}x^4 - \left( -\frac{1}{2} \cdot \frac{1}{4}x^4 \right) \right) \, dx \\
&= \int_{-1}^{1} x^4 \, dx \\
&= \left[ \frac{1}{5} x^5 \right]_{-1}^{1} \left( I \right) = \frac{2}{5}
\end{align*}
\]

**Example 5**

Find \( \iint_{\mathbb{R}^2} g(x,y) \, dx \, dy \) where

\[ g(x,y) = \begin{cases} x^2 y & 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases} \]

Region D: \( 0 \leq y \leq 1 \)

For \( x < 0 \), \( x = y \); for \( x > 0 \), \( y = x \)

Check: \( (0,0.5) \in D \)

\[
I = \int_{0}^{1} \int_{0}^{y} x^2 y \, dx \, dy
\]

\[
= \int_{0}^{1} \left[ \frac{1}{3} x^3 y \right]_{x=y}^{y} \, dy
\]

\[
= \int_{0}^{1} \frac{2}{3} y^4 \, dy = \frac{2}{15}
\]
\[ \int_{-1}^{0} \int_{y}^{x} x^2 y \, dy \, dx + \int_{0}^{1} \int_{x}^{y} x^2 y \, dx \, dy \]

\[ = \int_{-1}^{0} \frac{1}{2} x^2 (1-x^2) \, dx + \int_{0}^{1} \frac{1}{2} x^2 (1-x^2) \, dx \]

\[ = \left[ \frac{1}{6} x^3 - \frac{1}{10} x^5 \right]_{-1}^{0} + \left[ \frac{1}{6} x^3 - \frac{1}{10} x^5 \right]_{0}^{1} = \frac{2}{15} \]