Midterm #1 Solutions

ECE 564/645 Spring, 2013

1) (a) A system is digital if it chooses from one of a discrete number of waveforms to send a collection of bits.

(b) Large. Useful, because that is where I cannot simulate

(c) $d_{min}^{8\text{-}PSK} = 2\sqrt{E_s}\sin(\pi/8) \approx 2\sqrt{E_s}\cdot 3/8$

$\Rightarrow (d_{min}^{8\text{-}PSK})^2 = 9/16\, E_s$

$P(E) = Q\left(\frac{\cancel{2}\sqrt{E_s}\cdot 3/\cancel{8}4}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{9E_s}{32N_0}}\right)$

$d_{min}^{8\text{-}QAM} = 2$

$E_s = \frac{1}{8}\left(4\cdot(3^2+1^2) + 4(1^2+1^2)\right)$

$= \frac{1}{8}(40+8)$

$= 6$

$(d_{min}^{8\text{-}QAM})^2 = 4 = \frac{6}{3/2} = 2/3\, E_s$

$P(E) = Q\left(\frac{\sqrt{2/3\, E_s}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{E_s}{3N_0}}\right)$

8-QAM is $\frac{2/3\, E_s}{9/16\, E_s} = \frac{32}{27}$ better

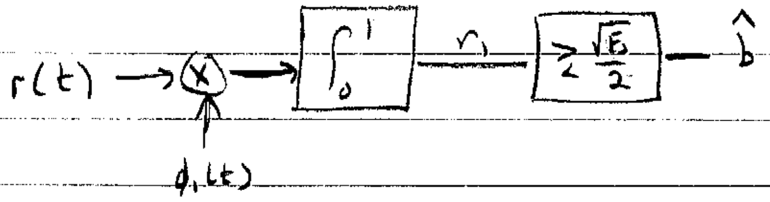
$10\log_{10}\frac{32}{27}$ dB better

2) (a)

clearly $\phi_1(t) = p(t)$ is a sufficient basis with

$$s_0(t) = 0 \cdot \phi_1(t)$$

$$s_1(t) = \sqrt{2E_s} \cdot \phi_1(t)$$



(b)

For binary

$$P_b = P(\epsilon) = Q\left(\frac{d}{\sqrt{2N_0}}\right) = Q\left(\frac{\sqrt{2E_s}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

(c) $10 \log_{10} 2 = 3$ dB worse than

$$Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

3)(a)

I claim

$$\phi_1(t) = s_1(t)$$

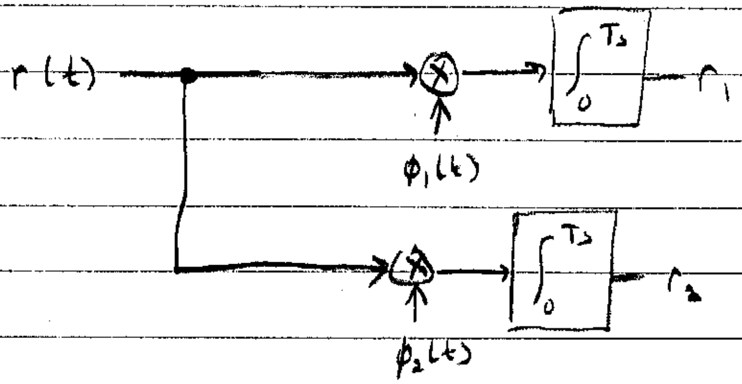
$$\phi_2(t) = s_3(t)/3$$

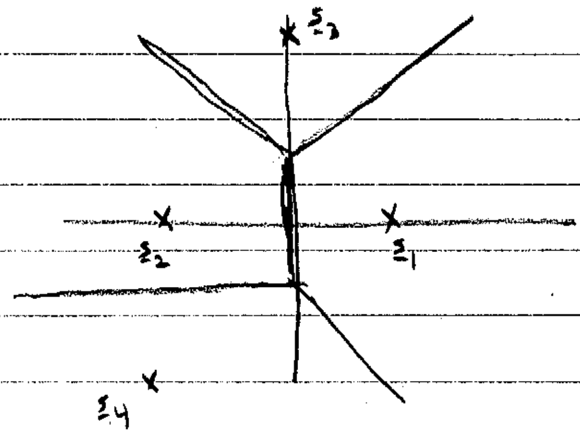
is a sufficient basis. Why? Clearly, $N \geq 2$ (since $s_3(t) \neq c\, s_1(t)$ for some c). And my $\phi_1(t)$ and $\phi_2(t)$ are orthonormal.

$$\underline{s}_1 = (1,0)^T \qquad \underline{s}_3 = (0,3)^T$$

$$\underline{s}_2 = (-1,0)^T \qquad \underline{s}_4 = (-1,-1)^T$$

(b)





(c)

$$P(E) = \sum_{i=1}^{m} P(E|\underline{s}_i)\, P(\underline{s}_i)$$

$$\leq \frac{1}{4} \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} Q\left(\frac{|\underline{s}_i - \underline{s}_j|}{\sqrt{2N_0}} \right)$$

distances

$i \backslash j$	1	2	3	4
1	-	2	$\sqrt{10}$	$\sqrt{5}$
2	2	-	$\sqrt{10}$	1
3	$\sqrt{10}$	$\sqrt{10}$	-	$\sqrt{17}$
4	$\sqrt{5}$	1	$\sqrt{17}$	-

$$P(E) \leq \frac{1}{4}\Bigg(\underbrace{2Q\left(\frac{1}{\sqrt{2N_0}}\right)}_{d_{ij}=1} + \underbrace{2Q\left(\sqrt{\frac{2}{N_0}}\right)}_{d_{ij}=2} + \underbrace{2Q\left(\sqrt{\frac{5}{2N_0}}\right)}_{d_{ij}=\sqrt{5}} + \underbrace{4Q\left(\sqrt{\frac{5}{N_0}}\right)}_{d_{ij}=\sqrt{10}}$$

$$+ \underbrace{2Q\left(\sqrt{\frac{17}{2N_0}}\right)}_{d_{ij}=\sqrt{17}} \Bigg)$$

(d)

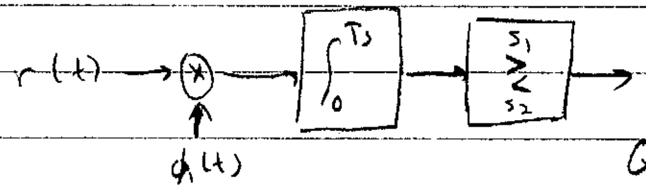
Clearly, choosing $\underline{s}_4$ given $\underline{s}_3$ (and vice versa) are covered by other events. You can remove

$$2Q\left(\sqrt{\frac{17}{2N_0}}\right)$$

4) (a)

$$\phi_1(t) = \sqrt{\frac{2}{T_s}}\cos(2\pi f_c t), \quad 0 \le t \le T_s$$

$$\underline{s}_1 = \sqrt{P_c T_s} \qquad \underline{s}_2 = -\sqrt{P_c T_s}$$
$$= \sqrt{E_s} \qquad\qquad = -\sqrt{E_s}$$



BPSK!

$$Q\left(\frac{d}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

(b)

New signal points:

$$\underline{s}_1 = \int_0^{T_s} \sqrt{2P_c}\cos(2\pi f_c t + \theta_\varepsilon)\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t)\,dt$$

$$= 2\sqrt{\frac{P_c}{T_s}}\int_0^{T_s}\Big(\cancel{\tfrac{1}{2}\cos(4\pi f_c t + \theta_\varepsilon)} + \tfrac{1}{2}\cos\theta_\varepsilon\Big)\,dt \quad \text{LPF}$$

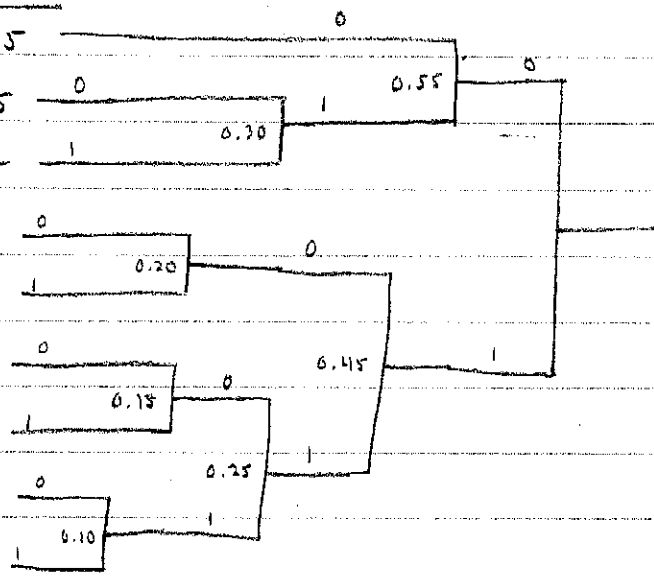
$$= 2\cdot\tfrac{1}{2}\sqrt{P_c T_s}\cos\theta = \sqrt{E_s}\cos\theta$$

$$\underline{s}_2 = \ldots = -\sqrt{E_s}\cos\theta$$

$$P(E) = Q\left(\frac{|\underline{s}_1 - \underline{s}_2|}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{E_s\cos^2\theta}{N_0}}\right)$$

5) (a)

$\underline{x}_i$	$P(\underline{X}_i = \underline{x}_i)$	code
(0,0)	0.25	00
(0,1)	0.15	010
(1,0)	0.15	011
(1,2)	0.10	100
(2,1)	0.10	101
(1,1)	0.09	1100
(2,1)	0.06	1101
(1,2)	0.06	1110
(2,2)	0.04	1111



$$R = 2(0.25) + 3(0.15 + 0.15 + 0.10 + 0.10) + 4(0.09 + 0.06 + 0.06 + 0.04)$$

$$= 3.0 \text{ bits}/_{2 \text{ symbols}} = 1.5 \text{ bits}/_{\text{symbol}}$$

$$1.485 = H(X) \le 1.5 \le H(X) + 1/2 = 1.985$$

[<u>Note</u>: $H(X_i) = -0.5 \log_2 0.5 - 0.3 \log_2 0.3 - 0.2 \log_2 0.2$

$= 0.5 + 0.52 + 0.46$

$= 1.485$]

(b) Thus, it is easy to get the bounds for the Huffman code (N=8) as:

$$1.485 = H(X_i) \le R_{Huff, N=8} \le H(X_i) + 1/N = 1.610$$

But an N=8 Huffman code is at least as good as the N=2 Huffman code.. why? I could use the N=2

Huffman code concatenated with itself (4 times) to get a lossless code of rate 1.5 bits/symbol, $N=8$, and we know the $N=8$ Huffman code is at least as good. Thus,

$$1.485 \le R_{HUFF, N=8} \le 1.5$$

are the tightest bounds.