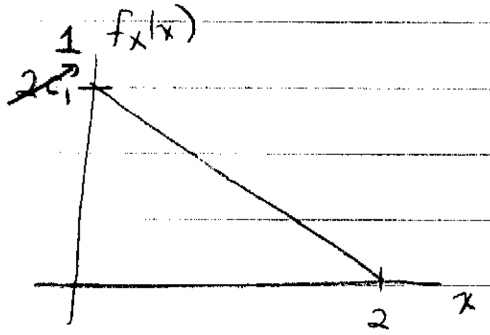
Midterm 2 Solutions

ECE 314 — Spring 2013

1)

(a)



Many ways

- $A_\Delta = \frac{1}{2}\cdot 2 \cdot 2c_1 \Rightarrow c_1 = \frac{1}{2}$
- $$\int_0^2 c_1(2-x)\,dx = c_1\left(2x - \tfrac{1}{2}x^2\right)\Big|_0^2 = c_1(4-2) = 2c_1 \Rightarrow c_1 = \tfrac{1}{2}$$

(b)

$$P(X \geq 1) = \int_1^2 \tfrac{1}{2}(2-x)\,dx = \tfrac{1}{2}\left(2x - \tfrac{1}{2}x^2\right)\Big|_1^2$$
$$= \tfrac{1}{2}(4-2) - \tfrac{1}{2}\left(2 - \tfrac{1}{2}\right)$$
$$= 1 - \tfrac{3}{4} = \tfrac{1}{4} \qquad \left(\text{Also } A_\Delta = \tfrac{1}{2}\cdot \overset{\text{base}}{1} \cdot \overset{\text{height}}{\tfrac{1}{2}} = \tfrac{1}{4}\right)$$

(c)

$$F_Y(y) \overset{0 \leq y \leq 4}{=} P(Y \leq y)$$
$$= P(X^2 \leq y)$$
$$= P(-\sqrt{y} \leq X \leq \sqrt{y})$$
$$= \int_0^{\sqrt{y}} \tfrac{1}{2}(2-x)\,dx$$
$$= \tfrac{1}{2}\left(2x - \tfrac{1}{2}x^2\right)\Big|_0^{\sqrt{y}}$$
$$= \tfrac{1}{2}\left(2\sqrt{y} - \tfrac{1}{2}y\right)$$
$$= \sqrt{y} - \tfrac{1}{4}y$$

Checks:
- $F_Y(0) = 0$
- $F_Y(4) = 1$

$$\Rightarrow f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} - \frac{1}{4}, & 0 \leq y \leq 4 \\ 0, & \text{else} \end{cases}$$

(d) • Lots of ways: I would roll the 10-sided die repeatedly for digits

0. _ _ _ _ _ ...

• Need the CDF:

$$F_X(x) = \int_0^x \tfrac{1}{2}(2-u)\,du$$

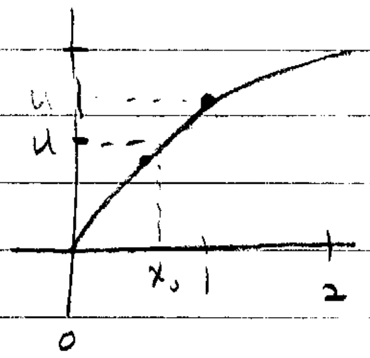
$$= \tfrac{1}{2}\left(2u - \tfrac{1}{2}u^2\right)\Big|_0^x$$

$$= \tfrac{1}{2}\left(2x - \tfrac{1}{2}x^2\right)$$

$$= x - \tfrac{1}{4}x^2$$

Checks:

- $x=0$, $F_X(x)=0$
- $x=2$, $F_X(2)=1$



Now, solve $u = x_0 - \tfrac{1}{4}x_0^2$ for x_0

$$x_0^2 - 4x_0 + 4u = 0$$

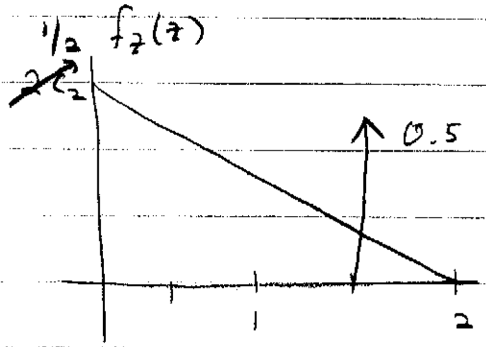
$$x_0 = \frac{4 \pm \sqrt{16-16u}}{2} = 2 \pm \sqrt{4-4u}$$

Want "$-$" (smaller root)

$$x_0 = 2 - \sqrt{4-4u} \qquad g(t) = 2 - \sqrt{4-4t}$$

2)

(a)



triangle must integrate to 0.5

$$\int_0^2 C_2(2-x)\,dx = 1/2$$

$$\Rightarrow C_2 = 1/4$$ (follow steps from 1(a))

(b)

$$E[Z^2] = \int_0^2 z^2\left(0.5\,\delta(z-3/2) + 1/4(2-z)\right)dz$$

$$= \int_0^2 0.5\,z^2\,\delta(z-3/2)\,dz$$
$$+ \int_0^2 \left(1/2\,z^2 - 1/4\,z^3\right)dz$$

$$= 0.5\cdot(3/2)^2 + 1/6\,z^3\Big|_0^2 - 1/16\,z^4\Big|_0^2$$

$$= 9/8 + 8/6 - 1$$

$$= 27/24 + 32/24 - 24/24 = 35/24$$

(c) First, note range$(V) = \{1/5, 3/5, 1, 7/5, 9/5\}$
<u>discrete</u>

Thus, just need probability of each V_k in the range.

$$p_V(1/5) = P(V=1/5) = \int_0^{2/5} 1/4(2-z)\,dz$$

$$P(V=3/5) = \int_{2/5}^{4/5} 1/4(2-z)\,dz$$

$$P(V=1) = \int_{4/5}^{6/5} 1/4(2-z)\,dz$$

$$P(V=7/5) = \int_{6/5}^{8/5}\left(\cancel{\delta(z-3/2)}\cdot 0.5 + 1/4(2-z)\right)dz$$

$$\xrightarrow{} 0.5 = 0.5 + \int_{6/5}^{8/5} 1/4(2-z)\,dz$$

$$P(V=9/5) = \int_{8/5}^{2} 1/4(2-z)\,dz$$

And then the CDF

$$F_V(v) = \begin{cases} 0, & v < 1/5 \\ p_V(1/5), & 1/5 \le v < 3/5 \\ p_V(1/5) + p_V(3/5), & 3/5 \le v < 1 \\ p_V(1/5) + p_V(3/5) + p_V(1), & 1 \le v < 7/5 \\ p_V(1/5) + p_V(3/5) + p_V(1) + p_V(7/5), & 7/5 \le v < 9/5 \\ 1, & v \ge 9/5 \end{cases}$$

3) (a) X is Gaussian, mean 2 and variance 4

$$P(-1 < X < 1) = \underbrace{P(X \le 1)}_{\frac{x-\mu}{\sigma} = \frac{1-2}{2} = -1/2} - \underbrace{P(X \le -1)}_{\frac{x-\mu}{\sigma} = \frac{-1-2}{2} = -3/2}$$

$$= 1 - \Phi(1/2) - (1 - \Phi(3/2))$$

$$= \Phi(3/2) - \Phi(1/2)$$

law of total probability

(b) $$P(-1 < Z < 1) = P(-1 < X < 1) \cdot 1/2$$

$$+ \underbrace{P(-1 < Y < 1)}_{P(Y \le 1) - P(Y \le -1)} \cdot 1/2$$

$$\frac{y-\mu}{\sigma} = \frac{1-(-2)}{4} = 3/4 \qquad \frac{y-\mu}{\sigma} = \frac{-1-(-2)}{4} = 1/4$$

$$= 1/2 \left(\Phi(3/2) - \Phi(1/2)\right)$$

$$+ 1/2 \left(\Phi(3/4) - \Phi(1/4)\right)$$

(c) $$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$$

$$= 1 - \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$$

$v = u/\sqrt{2}$

$dv = du/\sqrt{2}$

$$\overset{\downarrow}{=} 1 - \int_{u=\sqrt{2}x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/4} du/\sqrt{2}$$

$$= 1 - \frac{1}{2\sqrt{\pi}} \int_{\sqrt{2}x}^{\infty} e^{-u^2/4} du$$

$$= 1 - \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{\sqrt{\pi}/2} \int_{\sqrt{2}x}^{\infty} \frac{\sqrt{\pi}}{2} e^{-u^2/4} du = 1 - \frac{1}{\pi} A(\sqrt{2}x)$$

4) (a) $c/9 + 2c/9 + 3c/9 + 2c/9 = 1 \quad 8c = 9 \Rightarrow c = 9/8$

$$p_X(x_k) = \begin{cases} 1/8, & x_k = 2 \\ 1/4, & x_k = 3 \\ 3/8, & x_k = 4 \\ 1/4, & x_k = 5 \\ 0, & \text{else} \end{cases}$$

(b)

y \ x	2	3	4	5
1	1/24	0	0	0
2	1/24	1/12	0	0
3	1/24	1/12	1/8	0
4	0	1/12	1/8	1/12
5	0	0	1/8	1/12
6	0	0	0	1/12

method:

$p_{X,Y}(x_k, y_k)$

$= p_{Y|X}(y_k | x_k)\, p_X(x_k)$

(c) sum rows

$$p_Y(y_k) = \begin{cases} 1/24, & y_k = 1 \\ 1/24 + 1/12 = 1/8, & y_k = 2 \\ 1/4, & y_k = 3 \\ 7/24, & y_k = 4 \\ 5/24, & y_k = 5 \\ 1/12, & y_k = 6 \\ 0, & \text{else} \end{cases}$$

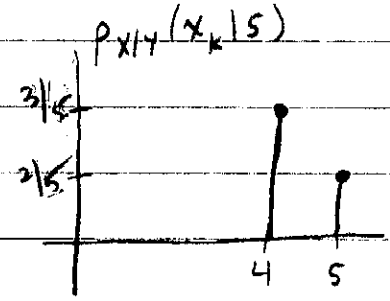
(d)

- $p_{X|Y}(x_k|5) = \dfrac{p_{X,Y}(x_k, 5)}{p_Y(5)}$

$$= \begin{cases} 3/5, & x_k = 4 \\ 2/5, & x_k = 5 \\ 0, & \text{else} \end{cases}$$

- choose most probable

$\hat{X} = 4$



(e)

No. $p_{X|Y}(x_k|5) \neq p_X(x_k)$.