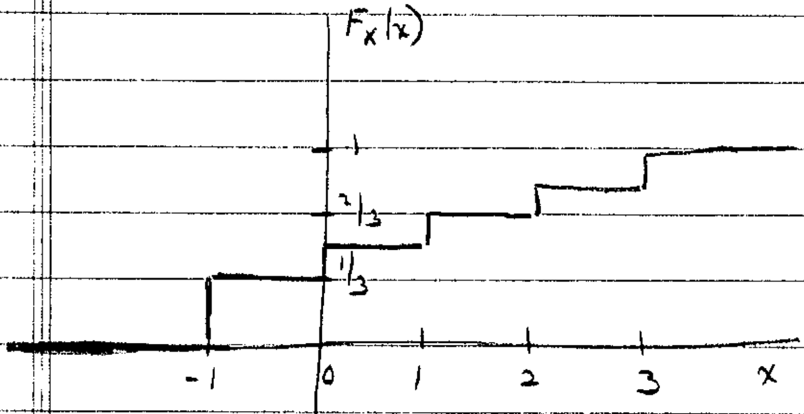
Midterm #1 Solutions

ECE 314 Spring, 2013

1) (a)

$$\sum_{x_k} p_X(x_k) = 1 \Rightarrow 3c = 1 \Rightarrow c = 1/3$$

(b) Jump by $p_X(x_k)$ at each x_k for which $p_X(x_k) \neq 0$.



$$F_X(x) = \begin{cases} 0, & x < -1 \\ 1/3, & -1 \le x < 0 \\ 1/2, & 0 \le x < 1 \\ 2/3, & 1 \le x < 2 \\ 5/6, & 2 \le x < 3 \\ 1, & 3 \le x \end{cases}$$

(c) $P(X \le 2) = 1 - P(X > 2) = 1 - P(X = 3) = 5/6$

-or-

$P(X \le 2) = F_X(2) = 5/6$

-or-

$P(X \le 2) = p_X(-1) + p_X(0) + p_X(1) + p_X(2) = 5/6$

(d) $P(X=0 \mid X \le 2) = \dfrac{P(\{X=0\} \cap \{X \le 2\})}{P(X \le 2)} = \dfrac{P(X=0)}{P(X \le 2)} = \dfrac{1/6}{5/6} = 1/5$

(e) $P(X^2 < 2) = p_X(-1) + p_X(0) + p_X(1) = 2/3$

(f) range$(Y) = \{0, 1, 4, 9\}$

$p_Y(0) = P(X < 1) = p_X(-1) + p_X(0) = 1/2$

$p_Y(1) = P(X = 1) = 1/6$

$p_Y(4) = P(X = 2) = 1/6$

$p_Y(9) = P(X = 3) = 1/6$

0 else

2)

(a) Bernoulli trials:

$$P(X=3) = \binom{12}{3} 0.5^3 \, 0.5^{12-3} = \binom{12}{3} 0.5^{12}$$

(b) Counting, unordered without replacement

"heads"

"tails"

$$P(A) = \frac{|A|}{|S|} = \frac{\binom{15}{3}\binom{15}{9}}{\binom{30}{12}}$$

(c) ~~P(third head on 12th draw)~~

$= P(\{\text{two heads in 11 draws}\} \cap \{12^{th} \text{ draw is head}\})$

~~$= P(\{12^{th} \text{ draw is head}\} \mid \{\text{two heads in 11 draws}\})$~~

$P(\{2 \text{ heads in 11 draws}\})$

heads left

$$= \frac{13}{19} \cdot \frac{\binom{15}{2}\binom{15}{9}}{\binom{30}{11}}$$

pieces of paper left

<u>smaller</u> set in (c) is a subset of set in (b).

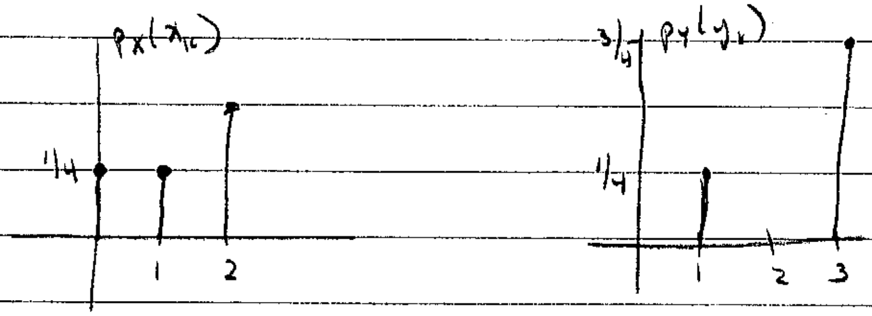
(d) Think "6" and "not 6" $\Rightarrow$ Bernoulli!

$$P(X=3) = \binom{12}{3}(1/6)^3(5/6)^9$$

(e) Need nine straight "not 6" $(5/6)^9$

3)

(a) Draw the pmfs:



Want outcomes to the left-small! $\underline{X}$.

(b) B: event shot closer than 1.5

$$P(B) = P(B|X)P(X) + P(B|Y)P(Y)$$

$$= (p_X(0) + p_X(1)) \cdot 1/2 + p_X(1) \cdot 1/2$$

$$= 1/2 \cdot 1/2 + 1/4 \cdot 1/2 = 3/8$$

(c) $$P(X|B) = \frac{P(B|X)P(X)}{P(B)} = \frac{1/2 \cdot 1/2}{3/8} = 2/3$$

(d) $\text{range}(Z) = \{0+1, 0+3, 1+1, 1+3, 2+1, 2+3\} = \{1, 2, 3, 4, 5\}$

$P(Z=1) = P(X=0) \cdot P(Y=1) = 1/4 \cdot 1/4 = 1/16$

$P(Z=2) = P(X=1) \cdot P(Y=1) = 1/4 \cdot 1/4 = 1/16$

$P(Z=3) = P(X=2) \cdot P(Y=1) + P(X=0) \cdot P(Y=3) = 1/2 \cdot 1/4 + 1/4 \cdot 3/4 = 5/16$

$P(Z=4) = P(X=1) \cdot P(Y=3) = 1/4 \cdot 3/4 = 3/16$

$P(Z=5) = P(X=2)\, P(Y=3) = 1/2 \cdot 3/4 = 3/8$

0 else

(a)

X can be 0, 1, 2, or 3

$$P(X=k) = P(X=k \mid B)\,P(B) + P(X=k \mid G)\,P(G)$$
$$= \binom{3}{k}(1/2)^k(1/2)^{3-k}\cdot 1/2$$
$$+ \binom{3}{k}(1/4)^k(3/4)^{3-k}\cdot 1/2$$
$$= \binom{3}{k}\cdot 1/16 + \binom{3}{k}\, 3^{3-k}\cdot 1/128$$

(b) A: event first two bits are in error

$$P(B \mid A) = \frac{P(A \mid B)\,P(B)}{P(A)} = \frac{P(A \mid B)\,P(B)}{P(A \mid B)\,P(B) + P(A \mid G)\,P(G)}$$

$$= \frac{1/2 \cdot 1/2 \cdot 1/2}{1/2 \cdot 1/2 \cdot 1/2 + 1/4 \cdot 1/4 \cdot 1/2}$$

$$= \frac{1/8}{1/8 + 1/32} = 4/5$$

(c) C: event first n bits are in error

$$P(B \mid C) = \frac{P(C \mid B)\,P(B)}{P(C)} = \frac{(1/2)^n \cdot \cancel{1/2}}{(1/2)^n \cdot \cancel{1/2} + (1/4)^n \cdot \cancel{1/2}}$$

$$= \frac{1}{1 + (1/2)^n} \to 1 \text{ as } n \to \infty$$

Lots of errors favors bad channel; in the limit, certainly the bad channel.