HW 9 Solutions

ECE 314 Introduction to Probability and Random Processes Spring 2013

April 23, 2013

Problem 1

(a)

We can easily find the piecewise constant density of Y

$$f_Y(y) = \begin{cases} \frac{1}{4}, & |y| \leq 1 \\ \frac{1}{8}, & 1 < |y| \leq 3 \\ 0, & otherwise \end{cases}$$

The conditional probabilities of X given Y are

$$P\{X = +1|Y = y\} = \begin{cases} 0, & -3 \leq y < -1 \\ \frac{1}{2}, & -1 \leq y \leq +1 \\ 1, & +1 < y \leq +3 \end{cases}$$

$$P\{X = -1|Y = y\} = \begin{cases} 1, & -3 \leq y < -1 \\ \frac{1}{2}, & -1 \leq y \leq +1 \\ 0, & +1 < y \leq +3 \end{cases}$$

Thus the best MSE estimate is

$$g(Y) = E(X\Big|Y) = \begin{cases} -1, & -3 \leq Y < -1 \\ 0, & -1 \leq Y \leq +1 \\ +1, & +1 < Y \leq +3 \end{cases}$$

(b)

The optimal decoder is given by the MAP rule. The a positerioiri pmf of X was found in part (a). Thus the MAP rule reduces to

$$D(y) = \begin{cases} -1, & -3 \le y \le -1 \\ \pm 1, & -1 < y < +1 \\ +1, & +1 \le y \le +3 \end{cases}$$

Since either value can be chosen for D(y) in the center range of Y, a symmetrical decoder is sufficient, i.e.,

$$D(y) = \begin{cases} -1, & y < 0 \\ +1, & y \ge 0 \end{cases}$$

The probability of decoding error is

$$\begin{aligned} P\{d(Y) \neq X\} &= P\{X = -1, Y \ge 0\} + P\{X = 1, Y < 0\} \\ &= P\{X = -1|Y \ge 0\}P\{Y \ge 0\} + P\{X = 1|Y < 0\}P\{Y < 0\} \\ &= \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} \\ &= \frac{1}{4} \end{aligned}$$

Problem 2

The sum of two jointly Gaussian random variales is still a Gaussian random variable. Calculate mean and variance of Z.

$$E(X^2) = var(X) + E(X)^2 = 1 + 0 = 1$$

$$E(Y^2) = var(Y) + E(Y)^2 = 1/2 + 1 = 1.5$$

$$
\begin{aligned}
\rho &= \frac{E[(X - E(X))(Y - E(Y))]}{\sigma_X \sigma_Y} \\
&= E(XY - X)/1 \cdot \sqrt{1/2} \\
&= (E(XY) - E(X))/\sqrt{1/2} \\
&= 1/2
\end{aligned}
$$

So

$$E(XY) = \sqrt{2}/4$$

$$E(Z) = E(5X + 3Y + 6) = 5E(X) + 3E(Y) + 6 = 9$$

$$
\begin{aligned}
var(Z) &= E(Z^2) - E(Z)^2 \\
&= E((5X + 3Y + 6)^2) - 9^2 \\
&= E(25X^2 + 30XY + 60X + 36Y + 9Y^2 + 36) - 81 \\
&= 25E(X^2) + 30E(XY) + 60E(X) + 36E(Y) + 9E(Y^2) + 36 - 81 \\
&= 25 + 30 \cdot \sqrt{2}/4 + 36 + 9 \cdot 1.5 + 36 - 81 \\
&= 29.5 + 7.5\sqrt{2}
\end{aligned}
$$

Problem 3

(a)

$E[X_i] = 3.5, Var[X_i] = \sum_{k=1}^{6} \frac{1}{6}(k - 3.5)^2 \approx 2.916$
Total = W = $\sum_{i=1}^{100} X_i$
Since X_i are iid $\Rightarrow E[W] = nE[X_i] = 100 * 3.5 = 350$

$$Var[W] = nVar[X_i] = 100 * 2.916 = 291.6$$

$$\begin{aligned}
P\Big(316 \le \sum_{i=1}^{100} X_i \le 384\Big) &= P\Big(\sum_{i=1}^{100} X_i \le 384\Big) - P\Big(\sum_{i=1}^{100} X_i \le 316\Big) \\
&= P\Big(\sum_{i=1}^{100} X_i - 350 \le 34\Big) - P\Big(\sum_{i=1}^{100} X_i - 350 \le -34\Big) \\
&= P\left(\frac{\sum_{i=1}^{100} X_i - 350}{\sqrt{291.6}} \le \frac{34}{\sqrt{291.6}}\right) - P\left(\frac{\sum_{i=1}^{100} X_i - 350}{\sqrt{291.6}} \le \frac{-34}{\sqrt{291.6}}\right) \\
&\approx \Phi\left(\frac{34}{\sqrt{291.6}}\right) - \Phi\left(\frac{-34}{\sqrt{291.6}}\right) \qquad \text{(CLT)} \\
&= \Phi(1.99) - \Phi(-1.99) \\
&= 0.95
\end{aligned}$$

(b)

Let $X_i = 1$ (i.e. i^{th} bit in error) and $X_i = 0$ (i.e. i^{th} bit not in error).
$P(X_i = 1) = 0.15$ and $P(X_i = 0) = 0.85$.
$E[X_i] = 1(.15) + 0(.85) = 0.15$
$E[X_i^2] = 0.15$
$Var[X_i] = 0.15 - (0.15)^2 = 0.1275$
Let $W = \sum_{i=1}^{100} X_i =$ number of bit errors.
$E[W] = nE[X_i] = 15, Var[W] = nVar[X_i] = 12.75$
You want $P\Big(\sum_{i=1}^{100} X_i > 20\Big)$

$$
\begin{aligned}
P\Big(\sum_{i=1}^{100} X_i > 20\Big) &= P\Big(\sum_{i=1}^{100} X_i - 15 > 5\Big) && \text{(subtract mean)} \\
&= P\left(\frac{\sum\limits_{i=1}^{100} X_i - 15}{\sqrt{12.75}} > \frac{5}{\sqrt{12.75}}\right) && \text{(divide by std dev)} \\
&= P(S_{100} > 1.40) \\
&= 1 - P(S_{100} \leq 1.40) \\
&\approx 1 - \Phi(1.40) && \text{(CLT)} \\
&= 0.0808
\end{aligned}
$$

Note, the exact solution is: $\sum\limits_{k=21}^{100} \binom{100}{k}(0.15)^k(0.85)^{100-k}$

(c)

Let X_i = number of messages during second i

W = number of total messages in a minute = $\sum\limits_{i=1}^{60} X_i$.

$\mu_x = E[X_i] = 10$, $\sigma_x^2 = E[X_i] = 10$ (Poisson Process)

$E[W] = 600$, $Var[W] = 600$

You want $P\Big(\sum\limits_{i=1}^{60} X_i > 650\Big)$.

$$
\begin{aligned}
P\Big(\sum_{i=1}^{60} X_i > 650\Big) &= P\left(\sum_{i=1}^{60} X_i - 600 > 50\right) \\
&= P\left(\frac{\sum\limits_{i=1}^{60} X_i - 600}{\sqrt{60(10)}} > \frac{50}{\sqrt{600}}\right) \\
&= P(S_{600} > 2.04) \\
&= 1 - P(S_{600} \leq 2.04) \\
&\approx 1 - \Phi(2.04) && \text{(CLT)} \\
&= 0.0207
\end{aligned}
$$

Problem 4

(a)

From the definitions:

$$\begin{aligned}
E(e^{jsV}) &= E(e^{jsXU}) \\
&= E_U(E_X(e^{jsXU}|U)) \\
&= E_X(e^{jsXU}|U=0)P(U=0) + E_X(e^{jsXU}|U=1)P(U=1) \\
&= E_X(e^{jsXU}|U=0)(1-\epsilon) + E_X(e^{jsXU}|U=1)(\epsilon) \\
&= (1-\epsilon) + (\epsilon)E_X(e^{jsX}) \\
&= (1-\epsilon) + (\epsilon)e^{js\mu_X}e^{-\frac{1}{2}\sigma^2 s^2} \\
&= (1-\epsilon) + (\epsilon)e^{-\frac{1}{2}\sigma^2 s^2}
\end{aligned}$$

(b)

From the definition of expectation:

$$E(V) = E(XU) = E(X)E(U) = 0$$

$$E(V^2) = E(X^2)E(U^2) = \epsilon\sigma^2$$

$$Var(V) = E(V^2) - E(V)^2 = \epsilon\sigma^2$$