ECE 314 - Introduction to Probability and Random Processes, Spring 2013
Homework #9
Due: 04/24/2013 (in lecture)

Exercise 1. *Estimation and detection*. Let the signal

$$X = \begin{cases} -1, & \text{with probability } \frac{1}{2} \\ +1, & \text{with probability } \frac{1}{2}. \end{cases}$$

and the noise $Z \sim \text{Unif}[-2, 2]$ be independent random variables. Their sum $Y = X + Z$ is observed.

(a) Find $E[X|Y]$ the best MSE estimate of X given Y.

(b) Now suppose we use a decoder to decide whether $X = +1$ or $X = -1$ so that the probability of error is minimized. This optimal decoder, called the Maximum a Posteriori (MAP) decoder, decides that $X = -1$ was transmitted if $P(X = -1|Y = y) > P(X = +1|Y = y)$, otherwise it will declare that $X = +1$ was transmitted. Find a numerical expression for the MAP decoder and its probability of error.

Exercise 2. The variables X and Y are jointly Gaussian and $X \sim \mathcal{N}(0, 1)$ and $Y \sim \mathcal{N}(1, 1/2)$. Also the correlation coefficient between X and Y is $\rho = 1/2$. Find the pdf of the random variable $Z = 5X + 3Y + 6$.

Exercise 3. Solve the following problems using the Central Limit Theorem:

(a) A fair die is tossed 100 times. Using the central limit theorem, estimate the probability that the total number of spots if between 316 and 384.

(b) A binary communication makes an error on each bit with probability 0.15. Estimate the probability that more than 20 bit errors are made in 100 bit transmissions. (Can you write down the exact expression for this probability?)

(c) The number of messages arriving per second at a switch in a computer network is a Poisson random variable with mean 10 messages/second. Estimate the probability that more than 650 messages arrive in one minute.

Exercise 4. Let X be a Gaussian random variable with $X \sim \mathcal{N}(0, \sigma^2)$ and let U be a Bernoulli random variable with $U \sim \text{Bern}(\epsilon)$ independent of X. Define V as $V = XU$.

(a) Find the characteristic function of V, $\varphi_V = E(e^{jsV}) = \int_{-\infty}^{+\infty} f_V(v)e^{jsv}\text{d}s$. *Hint:* use iterated expectation.

(b) Find the mean and variance of V.