HW 8 Solutions

ECE 314 Introduction to Probability and Random Processes Spring 2013

April 17, 2013

Problem 1

We can write the joint pdf for X and Y jointly Gaussian as:

$$f_{X,Y}(x,y) = \frac{exp(-[a(x-\mu_x)^2 + b(y-\mu_Y)^2 + c(x-\mu_x)(y-\mu_Y)])}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}}$$

where $a = \frac{1}{2(1-\rho_{X,Y}^2)\sigma_X^2}$, $b = \frac{1}{2(1-\rho_{X,Y}^2)\sigma_Y^2}$, $c = \frac{-2\rho_{X,Y}}{2(1-\rho_{X,Y}^2)\sigma_X\sigma_Y}$

By inspection of the given $f_{X,Y}(x,y)$ we find that: $a = \frac{2}{3}$, $b = \frac{8}{3}$, $c = \frac{4}{3}$, and we get three equations and three unknowns:

$$\begin{aligned} \rho_{X,Y} &= -\frac{c}{2\sqrt{ab}} = \frac{1}{2} \\ \sigma_X^2 &= \frac{1}{2(1-\rho_{X,Y}^2)a} = 1 \\ \sigma_Y^2 &= \frac{1}{2(1-\rho_{X,Y}^2)b} = \frac{1}{4} \end{aligned}$$

To find μ_X and μ_Y, we solve the equations:

$$2a\mu_X + c\mu_Y = 4$$
$$2b\mu_Y + c\mu_X = 8$$

and find that $\mu_X = 2$ and $\mu_Y = 1$. Finally:

$$Cov(X,Y) = \rho_{X,Y}\sigma_X\sigma_Y = -\frac{1}{4}$$

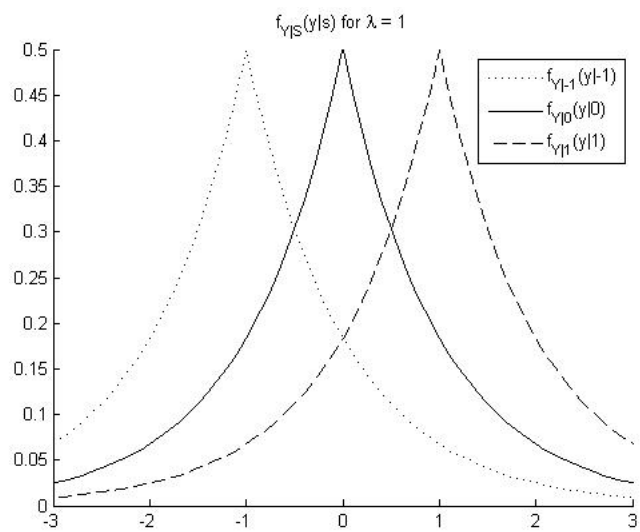
Problem 2

(a)

We use a trick here that is used several times in the lecture notes. Since $Y = S + Z$ and Z and S are independent, the conditional pdf is:

$$f_{Y|S}(y|s) = f_Z(y - s) = \frac{1}{2}\lambda e^{-\lambda|y-s|}$$

The plots are shown for $\lambda = 1$ in the figure below.



(b)

Inspection of the above figure shows how to calculate the probability of error.

$$
\begin{aligned}
P_e &= \sum_i \mathrm{P}\{\text{error}|i \text{ sent}\}\mathrm{P}\{i \text{ sent}\} = \frac{1}{3}\sum_i \mathrm{P}\{\text{error}|i \text{ sent}\} \\
&= \frac{1}{3}\left(1 - P\left\{-\frac{1}{2} < S + Z < +\frac{1}{2}\Big|S = 0\right\}\right) + \frac{1}{3}P\left\{S + Z > -\frac{1}{2}\Big|S = -1\right\} \\
&\quad +\frac{1}{3}P\left\{S + Z < +\frac{1}{2}\Big|S = +1\right\} \\
&= \frac{1}{3}(1 - P\left\{-\frac{1}{2} < Z < \frac{1}{2}\right\}) + \frac{1}{3}P\left\{Z < \frac{-1}{2}\right\} + \frac{1}{3}P\left\{Z > \frac{1}{2}\right\} \\
&= \frac{1}{3}P\left\{Z < \frac{-1}{2}\right\} + \frac{1}{3}P\left\{Z > \frac{+1}{2}\right\} + \frac{1}{3}P\left\{Z < \frac{-1}{2}\right\} + \frac{1}{3}P\left\{Z > \frac{1}{2}\right\} \\
&= \frac{2}{3}P\left\{Z < \frac{-1}{2}\right\} + \frac{2}{3}P\left\{Z > \frac{1}{2}\right\} \\
&= \frac{4}{3}P\left\{Z > \frac{1}{2}\right\} \text{ by symmetry} \\
&= \frac{4}{3}\int_{\frac{1}{2}}^{\infty} \frac{1}{2}\lambda e^{-\lambda|z|}dz \\
&= \frac{2}{3}e^{-\frac{1}{2}\lambda}
\end{aligned}
$$

Problem 3

(a)

$$\begin{aligned}
\rho(aX+b,Y) &= \frac{cov(aX+b,Y)}{\sqrt{var(aX+b)var(Y)}} \\
&= \frac{E\left[(aX+b-E[aX+b])(Y-E[Y])\right]}{\sqrt{a^2 var(X)var(Y)}} \\
&= \frac{E[(aX+b-aE[X]-b)(Y-E[Y])]}{a\sqrt{var(X)var(Y)}} \\
&= \frac{aE[(X-E[X])(Y-E[Y])]}{a\sqrt{var(X)var(Y)}} \\
&= \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}} \\
&= \rho(X,Y)
\end{aligned}$$

(b)

Look at the correlation coefficient of X and Y. Here, we have $X+Y=n$, and also $E[X]+E[Y]=n$. This:

$$X-E[X] = -(Y-E[Y])$$

We will calculate the correlation coefficient of X and Y, and verify that it is indeed equal to -1. We have:

$$\begin{aligned}
cov(X,Y) &= E[(X-E[X])(Y-E[Y])] \\
&= -E[(X-E[X])^2] \\
&= -var(X)
\end{aligned}$$

Hence, the correlation coefficient is:

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}} = \frac{-var(X)}{\sqrt{var(X)var(X)}} = -1$$