Problem 1

We can write the joint pdf for $X$ and $Y$ jointly Gaussian as:

$$f_{X,Y}(x,y) = \frac{\exp\left(-\left[a(x - \mu_x)^2 + b(y - \mu_Y)^2 + c(x - \mu_x)(y - \mu_Y)\right]\right)}{2\pi\sigma_X\sigma_Y\sqrt{1 - \rho_{X,Y}^2}}$$

where $a = \frac{1}{2(1 - \rho_{X,Y}^2)}$, $b = \frac{1}{2(1 - \rho_{X,Y}^2)}$, $c = \frac{-2\rho_{X,Y}}{2(1 - \rho_{X,Y}^2)}$.

By inspection of the given $f_{X,Y}(x,y)$ we find that: $a = \frac{2}{3}$, $b = \frac{8}{3}$, $c = \frac{4}{3}$, and we get three equations and three unknowns:

$$\rho_{X,Y} = -\frac{c}{2\sqrt{ab}} = \frac{1}{2}$$

$$\sigma_X^2 = \frac{1}{2(1 - \rho_{X,Y}^2)a} = 1$$

$$\sigma_Y^2 = \frac{1}{2(1 - \rho_{X,Y}^2)b} = \frac{1}{4}$$

To find $\mu_X$ and $\mu_Y$, we solve the equations:

$$2a\mu_X + c\mu_Y = 4$$
$$2b\mu_Y + c\mu_X = 8$$

and find that $\mu_X = 2$ and $\mu_Y = 1$. Finally:

$$Cov(X,Y) = \rho_{X,Y}\sigma_X\sigma_Y = -\frac{1}{4}$$
Problem 2

(a)

We use a trick here that is used several times in the lecture notes. Since
\( Y = S + Z \) and \( Z \) and \( S \) are independent, the conditional pdf is:

\[
f_{Y|S}(y|s) = f_Z(y - s) = \frac{1}{2} \lambda e^{-\lambda |y-s|}
\]

The plots are shown for \( \lambda = 1 \) in the figure below.
Inspection of the above figure shows how to calculate the probability of error.

\[
P_e = \sum_i P\{\text{error} | i \text{ sent}\} P\{i \text{ sent}\} = \frac{1}{3} \sum_i P\{\text{error} | i \text{ sent}\}
\]

\[
= \frac{1}{3} \left(1 - P\left\{-\frac{1}{2} < S + Z < +\frac{1}{2} | S = 0\right\}\right) + \frac{1}{3} P\left\{S + Z > -\frac{1}{2} | S = -1\right\}
\]

\[
+ \frac{1}{3} P\left\{S + Z < +\frac{1}{2} | S = +1\right\}
\]

\[
= \frac{1}{3} (1 - P\left\{-\frac{1}{2} < Z < +\frac{1}{2}\right\}) + \frac{1}{3} P\left\{Z < -\frac{1}{2}\right\} + \frac{1}{3} P\left\{Z > \frac{1}{2}\right\}
\]

\[
= \frac{1}{3} P\left\{Z < -\frac{1}{2}\right\} + \frac{2}{3} P\left\{Z > \frac{1}{2}\right\}
\]

\[
= \frac{4}{3} P\left\{Z > \frac{1}{2}\right\} \text{ by symmetry}
\]

\[
= \frac{4}{3} \int_{\frac{1}{2}}^{\infty} \frac{1}{2} e^{-\lambda |z|} dz
\]

\[
= \frac{2}{3} e^{-\frac{1}{2}\lambda}
\]
Problem 3

(a)

\[ \rho(aX + b, Y) = \frac{\text{cov}(aX + b, Y)}{\sqrt{\text{var}(aX + b) \text{var}(Y)}} \]

\[ = \frac{E[(aX + b - E[aX + b])(Y - E[Y])]}{\sqrt{a^2 \text{var}(X) \text{var}(Y)}} \]

\[ = \frac{E[(aX + b - aE[X] - b)(Y - E[Y])]}{a\sqrt{\text{var}(X) \text{var}(Y)}} \]

\[ = \frac{aE[(X - E[X])(Y - E[Y])]}{a\sqrt{\text{var}(X) \text{var}(Y)}} \]

\[ = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X) \text{var}(Y)}} \]

\[ = \rho(X, Y) \]

(b)

Look at the correlation coefficient of \( X \) and \( Y \). Here, we have \( X + Y = n \), and also \( E[X] + E[Y] = n \). This:

\[ X - E[X] = -(Y - E[Y]) \]

We will calculate the correlation coefficient of \( X \) and \( Y \), and verify that it is indeed equal to \(-1\). We have:

\[ \text{cov}(X, Y) = E[(X - E[X])(Y - E[Y])] \]

\[ = -E[(X - E[X])^2] \]

\[ = -\text{var}(X) \]

Hence, the correlation coefficient is:

\[ \rho(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X) \text{var}(Y)}} = \frac{-\text{var}(X)}{\sqrt{\text{var}(X) \text{var}(X)}} = -1 \]