ECE 314 - Introduction to Probability and Random Processes, Spring 2013
Homework #7
Due: 04/10/2013 (in lecture)

Exercise 1. Random variables X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 3(x+y), & 0 < y < 1, 0 < x < 1-y. \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the marginal probability density functions of X and Y.

(b) Find $P(X \geq \frac{1}{2})$.

(c) Find $P(X \geq Y)$.

(d) Find $P(X > \frac{1}{4} | Y < \frac{1}{2})$.

(e) Determine the joint cumulative distribution function (CDF) of X and Y.

Exercise 2. An important thing to understand is the following set of statements:

"*If random variables X and Y are independent, they are uncorrelated. If random variables X and Y are uncorrelated and jointly Gaussian [to be defined below], they are independent.*"

This problem gently walks you through the nuances of this thinking.

(a) Let Θ be a continuous random variable uniformly distributed on $[0, 2\pi]$. Let $X = \cos\Theta$ and $Y = \sin\Theta$. Show that X and Y are uncorrelated but not independent. (*Hint: As part of the solution, you will need to find $E[X]$, $E[Y]$ and $E[XY]$. This should be pretty easy; if you find yourself trying to find $f_X(x)$ or $f_Y(y)$, you are doing this the (very) hard way.*)

(b) (*Note: This part is ridiculously easy, but I give it to you so that you have seen it - and hopefully remember it.*)

Two random variables X and Y are jointly Gaussian if they have the joint probability density function:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\rho^2)}\left\{\left(\frac{x-E[X]}{\sigma_X}\right)^2 - 2\rho\frac{(x-E[X])}{\sigma_X}\frac{(y-E[Y])}{\sigma_Y} + \left(\frac{y-E[Y]}{\sigma_Y}\right)^2\right\}\right)$$

where $E[X]$ is the mean of X, $E[Y]$ is the mean of Y, σ_X is the standard deviation of X, σ_Y is the standard deviation of Y, and ρ is called the "correlation coefficient". From this monster equation, it is possible to show that:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{1}{2}\left(\frac{x - E[X]}{\sigma_X}\right)^2\right)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{1}{2}\left(\frac{y - E[Y]}{\sigma_Y}\right)^2\right)$$

If X and Y are uncorrelated, then $\rho = 0$. Show that $\rho = 0$ implies, for jointly Gaussian X and Y, that X and Y are independent.

Exercise 3. Let $Y = X + Z$, where the signal $X \sim U[-1, 1]$ and noise $Z \sim \mathcal{N}(0, 1)$ are independent. Your instructor tells you that the function $g(y) = \mathrm{E}[\mathrm{sgn}(X)|Y]$ is the *minimum mean square error*, i.e. the function that minimizes

$$\mathrm{MSE} = \mathrm{E}[(\mathrm{sgn}(X) - g(Y))^2],$$

where

$$\mathrm{sgn}(x) = \begin{cases} -1, & x < 0 \\ +1, & x > 0 \end{cases}.$$

(a) Show that if X and Z are independent then $f_{Y|X}(y|x) = f_Z(y - x)$, for example by using the CDF $F_{Y|X}(y|x) = P(Y \leq y|X = x)$.

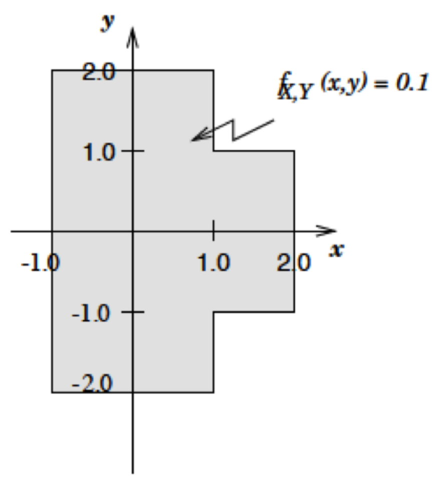
(b) Compute $g(y)$.

Exercise 4. Random variables X and Y have the joint PDF shown below (the $f_{X,Y}(x, y) = 1/10$ uniformly over the shaded gray area):

(a) Find the conditional PDF's $f_{Y|X}(y|x)$ and $f_{X|Y}(x|y)$, for all possible values of x and y, respectively.

(b) Find $E[X|Y = y]$ and $E[X]$.

(c) Find $E[Y|X = x]$ and $E[Y]$.


y
x
2.0
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-1.0
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2.0
$f_{X,Y}(x,y) = 0.1$