HW 6 Solutions

ECE 314 Introduction to Probability and Random Processes Spring 2013

March 29, 2013

Problem 1

We have for $x \geq 0$:

$$\begin{aligned} P(X > x|A) &= P(T > t + x|T > t) = \frac{P(T > t + x \text{ and } T > t)}{P(T > t)} \\ &= \frac{P(T > t + x)}{P(T > t)} = \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} = e^{-\lambda x} \end{aligned}$$

Problem 2

If $y = -t$ for $t < 0$ then $y \geq 0$. Because of the definition of g, random variable Y takes on non-negative values. Thus $f_Y(y) = 0$ for any negative y. For $y > 0$,

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(X\epsilon[-y, 0]) + P(X\epsilon(0, y^2]) \\ &= (F_X(0) - F_X(-y)) + (F_X(y^2) - F_X(0)) \end{aligned}$$

Taking the derivative of $F_Y(y)$ (and using the chain rule),

$$f_Y(y) = 2yf_x(y^2) + f_x(-y) = \frac{1}{\sqrt{2\pi}}(2ye^{-\frac{y^4}{2}} + e^{-\frac{y^2}{2}})$$

Problem 3

(a)

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_x(x) dx = \int_0^2 \frac{x^2}{2} dx \\ &= \left. \frac{x^3}{6} \right|_0^2 = 4/3 \end{aligned}$$

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f_x(x) dx = \int_0^2 \frac{x^3}{2} dx \\ &= \left. \frac{x^4}{8} \right|_0^2 = 2 \end{aligned}$$

$$E[X|\{X \geq 1\}] = \int_{-\infty}^{\infty} x f_{x|\{X\geq 1\}}(x) dx \quad (1)$$

Need to find $f_{x|\{X\geq 1\}}(x)$, first find $F_{X|\{X\geq 1\}}(x)$.
For $x \leq 1$, $F_{X|\{X\geq 1\}}(x) = 0$ and for $x \geq 2$, $F_{X|\{X\geq 1\}}(x) = 1$. Therefore for $1 \leq x \leq 2$:

$$F_{X|\{X\geq 1\}}(x) = \frac{P(\{X \leq x\} \cap \{X \geq 1\})}{P(X \geq 1)}$$

$$\text{where} \quad P(X \geq 1) = \int_1^2 \frac{x}{2} dx = \left. \frac{x^2}{4} \right|_1^2 = 3/4$$

$$\text{so: } F_{X|\{X\geq 1\}}(x) = \frac{\int_1^x \frac{u}{2} du}{3/4} = \frac{4}{3} \cdot \left. \frac{u^2}{4} \right|_1^x = \frac{x^2}{3} - \frac{1}{3}$$

and:

$$F_{X|\{X\geq 1\}}(x) = \begin{cases} 0, & x \leq 1 \\ x^2/3 - 1/3, & 1 \leq x \leq 2 \\ 1, & x \geq 2 \end{cases}$$

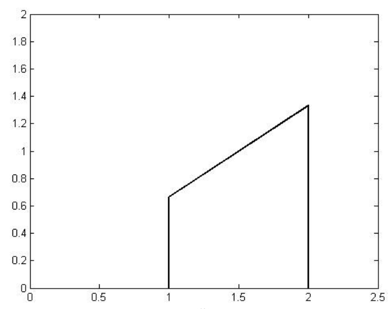
Using the fact that $f(x) = \frac{dF(x)}{dx}$:

$$f_{x|\{X\geq 1\}}(x) = \begin{cases} 2x/3, & 1 \leq x \leq 2 \\ 0, & else \end{cases}$$

Substitute the pdf above back into equation 1:

$$E[X|\{X \geq 1\}] = \int_1^2 x \cdot \frac{2x}{3} dx = \frac{2x^3}{9}\Big|_1^2 = 14/9$$

Note: the pdf integrates to 1.



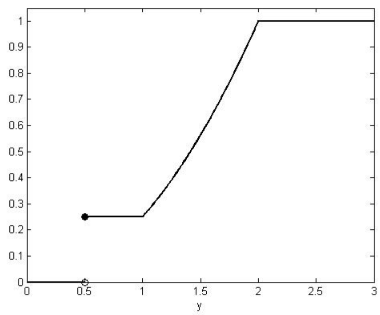
(b)

$F_Y(y) = P(Y \leq y)$. Since $Y \geq 0.5$, $F_Y(y) = 0$ for $y < 0.5$. What happens at $y = 0.5$?

$$P(Y = 0.5) = \int_0^1 \frac{x}{2} dx = \frac{x^2}{4}\Big|_0^1 = 1/4$$

For $y \geq 1$:

$$\begin{aligned} F_Y(y) = P(Y \leq y) &= \frac{1}{4} + \int_1^y \frac{x}{2} dx = \frac{1}{4} + \frac{x^2}{4}\Big|_1^y \\ &= \frac{1}{4} + \frac{y^2}{4} - \frac{1}{4} = \frac{y^2}{4} \end{aligned}$$



Plot of CDF:

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= 0.25\,\partial(y-0.5) + \begin{cases} y/2, & 1 \le y \le 2 \\ 0, & \text{else} \end{cases} \end{aligned}$$

(c)

$$\begin{aligned} E[Y] = \int_{\infty}^{\infty} y f_Y(y) dy &= 0.25 \cdot 0.5 + \int_1^2 y^2/2 dy \\ &= 1/8 + \left.\frac{y^3}{6}\right|_1^2 \\ &= 1/8 + (4/3 - 1/6) \\ &= 31/24 \end{aligned}$$

$$\begin{aligned} E[Y^2] = \int_{\infty}^{\infty} y^2 f_Y(y) dy &= 0.25(0.5)^2 + \int_1^2 y^3/2 dy \\ &= 1/16 + \left.\frac{y^4}{8}\right|_1^2 \\ &= 1/16 + (2 - 1/8) \\ &= 31/16 \end{aligned}$$

$$\begin{aligned} E[Y|\{X \ge 1\}] &= E[X|\{X \ge 1\}] \qquad \text{(since Y=X if } \{X \ge 1\}) \\ &= 14/9 \end{aligned}$$

Problem 4

(a)

$$P(N(t) \leq n) = \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

(b)

To find the pdf of $f_Y(y)$ of the random variable Y, note that the event $\{Y \leq t\}$ occurs iff (if and only if) the time of the n^{th} packet is in $[0, t]$. That is, iff the number $N(t)$ of packets arriving in $[0, t]$ is at least n. Alternatively, $\{Y > t\}$ occurs iff $\{N(t) < n\}$. Hence, the cdf of $F_Y(t)$ of Y is given by:

$$F_Y(t) = P(Y \leq t) = P(N(t) \geq n) = \sum_{k=n}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Differentiating $F_Y(t)$ with respect to t, we get the pdf $f_Y(t)$ as:

$$\begin{aligned}
f_Y(t) &= \sum_{k=n}^{\infty} \left[-\lambda e^{-\lambda t} \frac{(\lambda t)^k}{k!} + \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!} \right] \\
&= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} - \sum_{k=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^k}{k!} + \sum_{k=n+1}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!} \\
&= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}
\end{aligned}$$

for $t > 0$.