HW 5 Solutions

ECE 314 Introduction to Probability and Random Processes Spring 2013

March 12, 2013

Problem 1

(a)

Legitimate PMF because

$$\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots) = e^{-\lambda}e^{\lambda} = 1$$

(b)

$$\begin{aligned}
E[X] &= \sum_{k=0}^{\infty} ke^{-\lambda}\frac{\lambda^k}{k!} \\
&= \sum_{k=1}^{\infty} ke^{-\lambda}\frac{\lambda^k}{k!} \text{ the } k=0 \text{ term is zero} \\
&= \lambda\sum_{k=1}^{\infty} e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda\sum_{m=0}^{\infty} e^{-\lambda}\frac{\lambda^m}{m!} \text{ let } m=k-1 \\
&= \lambda
\end{aligned}$$

(c)

From the definition of Variance as Expectation of Square minus Square of Expectation:

$$\operatorname{var}(X) = E\left(X^2\right) - (E(X))^2$$

From Expectation of Function of Discrete Random Variable:

$$E\left(X^2\right) = \sum_{x \in \Omega_X} x^2 \Pr(X = x)$$

So:

$$E\left(X^2\right) = \sum_{k \geq 0} k^2 \frac{1}{k!} \lambda^k e^{-\lambda}$$ Definition of Poisson distribution

$$= \lambda e^{-\lambda} \sum_{k \geq 1} k \frac{1}{(k-1)!} \lambda^{k-1}$$ Note change of limit: term is zero when $k = 0$

$$= \lambda e^{-\lambda} \left(\sum_{k \geq 1} (k-1) \frac{1}{(k-1)!} \lambda^{k-1} + \sum_{k \geq 1} \frac{1}{(k-1)!} \lambda^{k-1} \right)$$ straightforward algebra

$$= \lambda e^{-\lambda} \left(\lambda \sum_{k \geq 2} \frac{1}{(k-2)!} \lambda^{k-2} + \sum_{k \geq 1} \frac{1}{(k-1)!} \lambda^{k-1} \right)$$ Again, note change of limit: term is zero when $k - 1 = 0$

$$= \lambda e^{-\lambda} \left(\lambda \sum_{i \geq 0} \frac{1}{i!} \lambda^i + \sum_{j \geq 0} \frac{1}{j!} \lambda^j \right)$$ putting $i = k - 2, j = k - 1$

$$= \lambda e^{-\lambda} \left(\lambda e^{\lambda} + e^{\lambda} \right)$$ Taylor Series Expansion for Exponential Function

$$= \lambda(\lambda + 1)$$
$$= \lambda^2 + \lambda$$

Then:

$$\operatorname{var}(X) = E\left(X^2\right) - (E(X))^2$$
$$= \lambda^2 + \lambda - \lambda^2$$ Expectation of Poisson Distribution: $E(X) = \lambda$
$$= \lambda$$

Problem 2

(a)

$x = 0$ maximizes $\mathrm{E}[Y|X = x]$ since

$$\mathrm{E}[Y|X = x] = \begin{cases} 2 & x = 0 \\ 3/2 & x = 2 \\ 3/2 & x = 4 \\ 0 & \text{otherwise.} \end{cases}$$

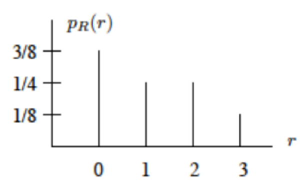
(b)

$y = 3$maximizes $\mathrm{Var}(X|Y = y)$ since

$$\mathrm{Var}[X|Y = y] = \begin{cases} 0 & y = 0 \\ 8/3 & y = 1 \\ 1 & y = 2 \\ 4 & y = 3 \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

(c)

Among the event pairs of(X, Y) the pairs(0,1),(0,3)and(4,0)give $r = 0$;(2,1)and(4,1)give $r = 1$;(3,2),(4,2)give $r = 2$;(4,3)give$r = 3$.



(d)

By traversing the points top to bottom and left to right, we obtain,

$$\mathrm{E}[XY] = \frac{1}{8}(0 \cdot 3 + 4 \cdot 3 + 2 \cdot 2 + 4 \cdot 2 + 0 \cdot 1 + 2 \cdot 1 + 4 \cdot 1 + 4 \cdot 0) = \frac{15}{4}.$$

Conditioning on A removes the point masses at $(0,1)$ and $(0,3)$. The conditional probability of the remaining point masses is thus $1/6$, and

$$\mathrm{E}[XY|A] = \frac{1}{6}(4\cdot 3 + 2\cdot 2 + 4\cdot 2 + 2\cdot 1 + 4\cdot 1 + 4\cdot 0) = 5.$$

Problem 5

(a)

Since $P(\tau = t) = 0$ (continuous random variable)
$F_\tau(t) = P(\tau \leq t) = 1 - P(\tau \geq t) = 1 - e^{-0.25t}$ for $t \geq 0$. Therefore we have,
$F_x(x) = \begin{cases} 0 & t < 0 \\ 1 - e^{-0.25t} & t \geq 0 \end{cases}$

(b)

$f_\tau(t) = \frac{d}{dt}F_\tau(t) = \begin{cases} 0 & t < 0 \\ 0.25e^{-0.25t} & t \geq 0 \end{cases}$

(c)

$P(\tau \geq 4.5) = \int_{4.5}^{\infty} 0.25e^{-0.25t}dt = -e^{-0.25t}|_{4.5}^{\infty} = e^{-0.25t\cdot 0.45}$

(d)

$P(2.5 \leq \tau \leq 4.5) = \int_{2.5}^{4.5} 0.25e^{-0.25t}dt = -e^{-0.25t}|_{2.5}^{4.5} = e^{-0.25t\cdot 2.5} - e^{-0.25t\cdot 4.5}$

(e)

$$\begin{aligned} P(\tau \geq 9|\tau \leq 11) &= \frac{P(\{\tau \geq 9\} \cap \{\tau \leq 11\})}{P(\{\tau \leq 11\})} \\ &= \frac{P(\{9 \leq \tau \leq 11\})}{P(\{\tau \leq 11\})} \\ &= \frac{\int_9^{11} 0.25e^{-0.25t}dt}{\int_0^{11} 0.25e^{-0.25t}dt} \\ &= \frac{e^{-0.25\cdot 9} - e^{-0.25\cdot 11}}{1 - e^{-0.25\cdot 11}} \end{aligned}$$

Problem 3

1) (a)

We know $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\int_{-\infty}^{\infty} c e^{-|x|}\,dx = c\left(\int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx\right)$$

$$= c\left(e^{x}\Big|_{-\infty}^{0} - e^{-x}\Big|_{0}^{\infty}\right)$$

$$= 2c$$

$$\Rightarrow c = 1/2$$

(b) $P(X \le 0) = \int_{-\infty}^{0} 1/2\, e^{x}\,dx = 1/2\, e^{x}\Big|_{-\infty}^{0} = 1/2$

(c) $P(-1 \le X \le 2) = \int_{-1}^{2} f_X(x)\,dx = \int_{-1}^{0} 1/2\, e^{x}\,dx + \int_{0}^{2} 1/2\, e^{-x}\,dx$

$$= 1/2\, e^{x}\Big|_{-1}^{0} - 1/2\, e^{-x}\Big|_{0}^{2}$$

$$= 1/2 - 1/2\, e^{-1} - 1/2\, e^{-2} + 1/2$$

$$= 1 - 1/2\, e^{-1} - 1/2\, e^{-2}$$

(d) $P(X^2 \ge 4) = P(\{X \le -2\} \cup \{X \ge 2\})$

$$\overset{\text{symmetry}}{=} 2\int_{2}^{\infty} 1/2\, e^{-x}\,dx = 2\cdot\left(-1/2\, e^{-x}\right)\Big|_{2}^{\infty} = e^{-2}$$

(e) $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$= \underbrace{\int_{-\infty}^{0} 1/2\, x e^{x}\,dx}_{u = -x} + \int_{0}^{\infty} 1/2\, x e^{-x}\,dx$$

$$= -\int_{0}^{\infty} 1/2\, u e^{-u}\,du + \int_{0}^{\infty} 1/2\, x e^{-x}\,dx$$

$$= 0$$

(f)

E$[X^2]$=$\int_{-\infty}^{\infty} x^2 f_X(x)dx$

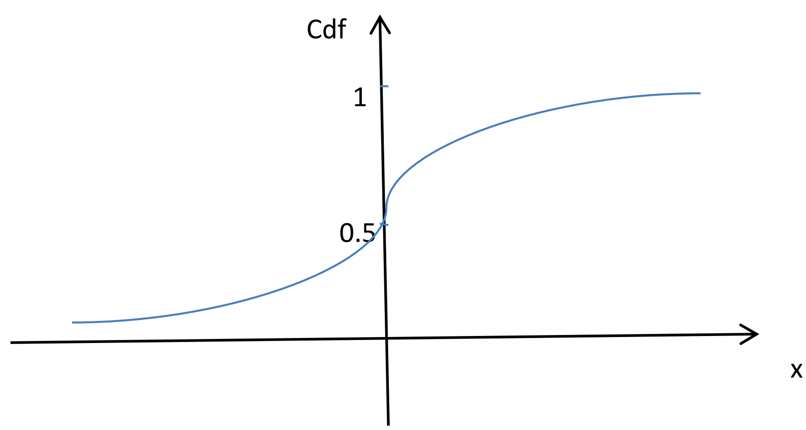
$$= \int_{-\infty}^{0} \frac{1}{2}x^2e^x dx + \int_{0}^{+\infty} \frac{1}{2}x^2e^{-x}dx = 2\int_{-\infty}^{0} \frac{1}{2}x^2e^x = \ [x^2e^x]\ \begin{matrix}0\\-\infty\end{matrix} - \int_{-\infty}^{0} 2xe^x dx = 2$$

(g)

Var(X)=E$[X^2]$-$E[X]^2$=2-0=2

(h)

$$F_x(x) = \int_{-\infty}^{x} f_x(x) = \begin{cases} \int_{-\infty}^{0} \frac{1}{2}e^x + \int_{0}^{x} \frac{1}{2}e^{-x} \ = 1 - \frac{1}{2}e^{-x} & for\ x \geq 0 \\ \int_{-\infty}^{x} \frac{1}{2}e^x \ = \frac{1}{2}e^x & for\ x < 0 \end{cases}$$



Problem 4

(a) The optimal MAP rule is equivalent to the ML rule

$$D(x) = \begin{cases} 0, & f_{X|\Theta}(x|0) > f_{X|\Theta}(x|1), \\ 1, & \text{otherwise.} \end{cases}$$

Since the Unif(0, 1) pdf $f_{X|\Theta}(x|0)$ is larger than the Exp(1) pdf $f_{X|\Theta}(x|1)$ for $0 < x < 1$, we have

$$D(x) = \begin{cases} 0, & 0 < x < 1, \\ 1, & \text{otherwise.} \end{cases}$$

(b) The probability of error is given by

$$\begin{aligned} \mathrm{P}(\Theta \neq D(X)) &= \frac{1}{2}\mathrm{P}(\Theta \neq D(X)|\Theta = 0) + \frac{1}{2}\mathrm{P}(\Theta \neq D(X)|\Theta = 1) \\ &= \frac{1}{2}\mathrm{P}(X > 1|\Theta = 0) + \frac{1}{2}\mathrm{P}(0 < X < 1|\Theta = 1) \\ &= \frac{1}{2}(1 - e^{-1}). \end{aligned}$$