HW 4 Solutions

ECE 314 Introduction to Probability and Random Processes Spring 2013

March 7 , 2013

Problem 1

To verify this rule, we let $Y = g(x)$ and use the formula
$p_Y(y) = \sum_{\{x|g(x)=y\}} p_X(x)$

$$\begin{aligned}
E[g(x)] &= E[Y] \\
&= \sum_y y p_Y(y) \\
&= \sum_y y \sum_{\{x|g(x)=y\}} p_X(x) \\
&= \sum_y \sum_{\{x|g(x)=y\}} y p_X(x) \\
&= \sum_y \sum_{\{x|g(x)=y\}} g(x) p_X(x) \\
&= \sum_x g(x) p_X(x)
\end{aligned}$$

Problem 2

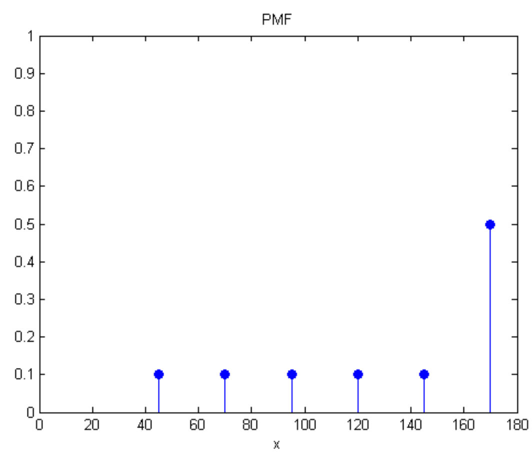
(a)

Here is the easiest way to approach this somewhat difficult problem

Figure out the possible costs
$S = \{45, 70, 95, 120, 145, 170\}$

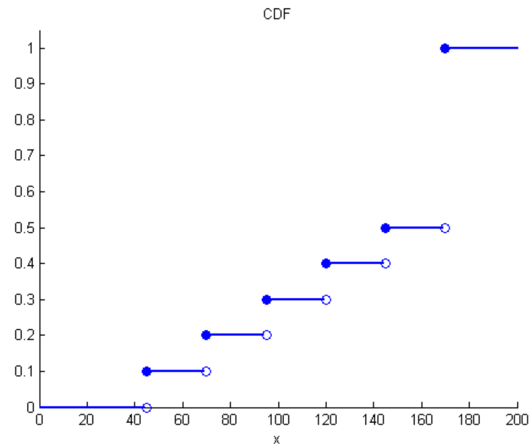
Now figure out the probability of each one:
$P(X = 45) = P(\tau = 1) = \frac{1}{10}$
$P(X = 70) = P(\tau = 2) = \frac{1}{10}$
$P(X = 95) = P(\tau = 3) = \frac{1}{10}$
$P(X = 120) = P(\tau = 4) = \frac{1}{10}$
$P(X = 145) = P(\tau = 5) = \frac{1}{10}$
$P(X = 170) = P(\tau = \{6, 7, 8, 9, 10\}) = \frac{1}{2}$

$$\Rightarrow p_X(x_k) = \begin{cases} 1/10, & x_k \epsilon \{45, 70, 95, 120, 145\} \\ 1/2, & x_k = 170 \\ 0, & \text{else} \end{cases}$$



(b)

$$F_X(x) = \begin{cases} 0, & x < 45 \\ 1/10, & 45 \le x < 70 \\ 2/10, & 70 \le x < 95 \\ 3/10, & 95 \le x < 120 \\ 4/10, & 120 \le x < 145 \\ 5/10, & 145 \le x < 170 \\ 1, & x \ge 170 \end{cases}$$



(c)

- $P(X \leq 70) = F_x(70) = 2/10 = 1/5$
- $P(X \leq 70) = P(\tau \leq 2) = 2/10 = 1/5$

(d)

$$\begin{aligned} E[X] &= \sum_{x_k \epsilon S} p_x(x_k) \\ &= \frac{1}{10} \cdot 45 + \frac{1}{10} \cdot 70 + \frac{1}{10} \cdot 95 + \frac{1}{10} \cdot 120 + \frac{1}{10} \cdot 145 + \frac{1}{2} \cdot 170 \\ &= 132.5 \end{aligned}$$

(e)

I know it will cost 145 or 170 dollars

$$\begin{aligned} P(\{X = 145\} | \{X \geq 145\}) &= \frac{P(\{X = 145\} \cap \{X \geq 145\})}{P(X \geq 145)} \\ &= \frac{P(X = 145)}{P(X \geq 145)} \\ &= \frac{1/10}{6/10} = 1/6 \end{aligned}$$

$$
\begin{aligned}
P(\{X = 170\}|\{x \geq 145\}) &= \frac{P(\{X = 170\} \cap \{X \geq 145\})}{P(X \geq 145)} \\
&= \frac{P(X = 170)}{P(X \geq 145)} \\
&= \frac{5/10}{6/10} = 5/6
\end{aligned}
$$

$$
\Rightarrow p_{x|x \geq 145}(x_k) = \begin{cases} 1/6, & x_k = 145 \\ 5/6, & x_k = 170 \\ 0, & \text{else} \end{cases}
$$

Problem 3

(a)

The joint pmf can be computed using the chain rule $p_{x,y} = p_{Y|X}(y|x)p_X(x)$:

$$p_{X,Y}(0,0) = \frac{9}{30}$$

$$p_{X,Y}(0,1) = 0$$

$$p_{X,Y}(0,2) = \frac{1}{30}$$

$$p_{X,Y}(1,0) = 0$$

$$p_{X,Y}(1,1) = \frac{16}{30}$$

$$p_{X,Y}(1,2) = \frac{4}{30}$$

The marginal pmf $p_Y(y)$ is found by the law of total probability $p_Y(y) = \sum_x p_{X,Y}(x,y)$:

$$p_Y(Y = 0) = \frac{9}{30}$$

$$p_Y(Y = 1) = \frac{16}{30}$$

$$p_Y(Y = 2) = \frac{5}{30}$$

Finally we use the definition of conditional probability, $p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$:

$$p_{X|Y}(0|0) = 1,\ p_{X|Y}(1|0) = 0$$

$$p_{X|Y}(0|1) = 0,\ p_{X|Y}(1|1) = 1$$

$$p_{X|Y}(0|2) = \frac{1}{5},\ p_{X|Y}(1|2) = \frac{4}{5}$$

Note that all three pmfs sum to 1.

(b)

An error occurs when $X = 0, Y = 2$ or $X = 1, Y = 2$. Then, the probability of error is:

$$P_e\{X \neq Y\} = p_{X,Y}(0,2) + p_{X,Y}(1,2) = \frac{5}{30}$$

problem 4

(a)

$$(1+1)c + (1+9)c + (4+1)c + (4+9)c + (16+1)c + (16+9)c = 1$$

$$c = \frac{1}{72}$$

(b)

$$P(Y < X) = P(\{2,1\}) + P(\{4,1\}) + P(\{4,3\}) = \frac{5}{72} + \frac{17}{72} + \frac{25}{72} = \frac{47}{72}$$

(c)

$$P(Y > X) = P(\{1,3\}) + P(\{2,3\}) = \frac{10}{72} + \frac{13}{72} = \frac{23}{72}$$

(d)

$$P(Y = X) = P(\{1,1\}) = \frac{2}{72}$$

(e)

$$P(Y=3) = P(\{1,3\}) + P(\{2,3\}) + P(\{4,3\}) = \frac{10}{72} + \frac{13}{72} + \frac{25}{72} = \frac{48}{72}$$

(f)

$P_X(x) = \sum_{y=-\infty}^{\infty}$ and $P_Y(y) = \sum_{x=-\infty}^{\infty}$

$$p_X(x) = \begin{cases} \frac{2}{72}, & x = 1 \\ \frac{18}{72}, & x = 2 \\ \frac{42}{72}, & x = 4 \\ 0, & otherwise \end{cases} \qquad p_Y(y) = \begin{cases} \frac{24}{72}, & y = 1 \\ \frac{48}{72}, & y = 3 \\ 0, & otherwise \end{cases}$$

(g)

$$E[X] = \sum_{x=-\infty}^{\infty} x p_X(x)$$

$$E[X] = 1 \times \frac{12}{72} + 2 \times \frac{18}{72} + 4 \times \frac{42}{72} = 3$$

$$E[Y] = 1 \times \frac{24}{72} + 3 \times \frac{48}{72} = \frac{7}{3}$$

(h)

There are four (x, y) coordinate pairs in A:$(1,1), (2,1), (4,1), (4,3)$. Therefore, $P(A) = \frac{1}{72}(2 + 5 + 17 + 25) = \frac{49}{72}$

$$p_{X|A}(x) = \begin{cases} \frac{2}{49}, & x = 1 \\ \frac{5}{49}, & x = 2 \\ \frac{42}{49}, & x = 4 \\ 0, & otherwise \end{cases} \qquad E[X|A] = 1 \times \frac{2}{49} + 2 \times \frac{5}{49} + 4 \times \frac{42}{49} = \frac{180}{49}$$