ECE 314 - Introduction to Probability and Random Processes, Spring 2013
Homework #4
Due: 03/06/2013 (in lecture)

Exercise 1. In class you have looked at expectations and you have learned some expressions without proof. This problem explores an easy mathematical proof. Derive the expected value rule for functions of random variables $\mathrm{E}[g(X)] = \sum_x g(x)p_X(x)$.

Hint: Let $Y = g(X)$ and use the formula $p_Y(y) = \sum_{\{x|g(x)=y\}} p_X(x)$.

Exercise 2. A service facility charges a \$20 fixed fee plus \$25 per hour of service up to 6 hours, and no additional fee is charged for service exceeding 6 hours. Suppose that the service time τ is equally likely to be any number of hours in $\{1, 2, 3, 4, \ldots, 10\}$ hours (Assume that it takes some integer number of hours). Let X represent the cost of service in the facility.

(a) Find **and sketch** the probability mass function for X.

(b) Find **and sketch** the cumulative distribution function for X.

(c) What is the probability that you end up paying less than or equal to \$70 for service? Answer this part two different ways:

- Using your answer to part (b).
- Finding the time (call it τ_0) at which the service would cost *exactly* \$70 and then finding the probability that $\tau \leq \tau_0$.

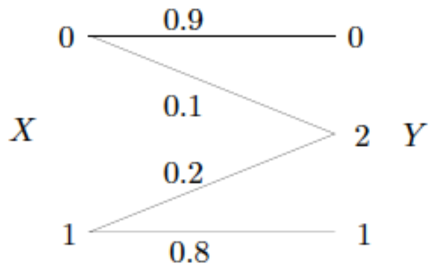
(d) What is the expected value (or mean) of X?

(e) The repair shop calls after 4.5 hours and tells you that your car is not ready yet; hence, this is going to cost you at least \$145. Given this information, come up with a new probability mass function for how much it will cost.

Exercise 3. *Binary erasure channel*. A binary erasure channel has binary input $X \in \{0, 1\}$ and ternary output $Y \in \{0, 1, 2\}$ (see the probability transition diagram below). The erasure, represented by 2, corresponds to the case when the receiver cannot determine whether the received signal is a 0 or a 1. When a 0 is sent, a 0 is received with probability 0.9, i.e., $p_{Y|X}(0|0) = 0.9$, and a 2 is received with probability 0.1. When a 1 is sent, a 1 is received with probability 0.8 and a 2 is received with probability 0.2. In answering the following questions, assume that the probability of transmitting a zero is $P(X = 0) = \frac{1}{3}$ and of transmitting a one is $\frac{2}{3}$ (we say that X is a Bernoulli random variable, i.e. $X \sim \text{Bern}(\frac{1}{3})$).

(a) Find $p_{X,Y}(x, y)$, $p_Y(y)$, and $p_{X|Y}(x|y)$ for $y = 0, 1, 2$.

(b) Find the error probability $P_e = P(X \neq Y)$.



Exercise 4. Random variables X and Y have the joint pmf (i.e. the pmf of the pair (X, Y))

$$p_{X,Y}(x_k, y_k) = \begin{cases} c(x_k^2 + y_k^2), & x_k \in \{1, 2, 4\} \text{ and } y_k \in \{1, 3\} \\ 0, & \text{otherwise.} \end{cases}$$

where c is some constant.

(a) What is the value of the constant c that makes the above a valid joint pmf of the pair (X, Y)?

(b) What is $P(Y < X)$?

(c) What is $P(Y > X)$?

(d) What is $P(Y = X)$?

(e) What is $P(Y = 3)$?

(f) Find the marginal pmf's $p_X(x_k)$ and $p_Y(y_k)$.

(g) Find the expectations $E[X], E[Y]$.

(h) Let A denote the event $\{X \geq Y\}$. Find $p_{X|A}(x_k)$ and $E[X|A]$.