Solutions homework 3

Problem 1

We break this into two parts.

The first part we look at is that the 15th flip is the 10th head. That means that the 15th flip is a head (this occurs with probability p) and 9 of the first 14 flips were heads. Since flipping a coin 14 times is a Bernoulli process we know this probability must be $p^9q^5C(14; 9)$ then the probability that the 15th flip is the 10th head is $p^{10}q^5C(14; 9)$.

The second part we look at is that the 15th flip is the 12th tail. That means that the 15th flip is a tail (this occurs with probability q) and 3 of the first 14 flips were heads(11 of them 111were tails and 11+3=14). Since flipping a coin 14 times is a Bernoulli process we know this probability must be $p^3q^{11}C(14; 3)$ then the probability that the 15th flip is the 12th tail is $p^3q^{12}C(14; 3)$.

So the probability that either of the two events happen is

$p^{10}q^5C(14; 9) + p^3q^{12}C(14; 3)$

Problem 2

\underline{Problem 1} \\

Let the event A be the event that the professor teaches her class, and let B be the event that the weather is bad. We have

\begin{displaymath}

P(A) = P(B)P(A|B) + P(\bar{B})P(A|\bar{B}),

\end{displaymath}

and

\begin{displaymath}

P(A|B)=\sum_{i=k}^n \binom{n}{k}p_b^i(1-p_b)^{n-i} ~~~~and~~~~~

P(A|\bar{B})=\sum_{i=k}^n \binom{n}{k}p_g^i(1-p_g)^{n-i}.

\end{displaymath}

Therefore,

\begin{displaymath}

P(A) = P(B)\sum_{i=k}^n \binom{n}{k}p_b^i(1-p_b)^{n-i} + (1-P(B))\sum_{i=k}^n \binom{n}{k}p_g^i(1-p_g)^{n-i}.

\end{displaymath}

Problem 3

(a) $k/2 + k/4 + k/2 + k/4 + k/2 = 2k$

$$\Rightarrow k = 1/2$$

(b) $P(X \geq 0) = P(X \in \{0,1,2\})$

$$\overset{\text{disjoint}}{=} P(X=0) + P(X=1) + P(X=2)$$

$$= 1/4 + 1/8 + 1/4 = 5/8$$

(c) $P(X \leq 1 \mid X \geq 0) = \dfrac{P(\{X \leq 1\} \cap \{X \geq 0\})}{P(X \geq 0)}$

$$= \frac{P(X \in \{0,1\})}{P(X \in \{0,1,2\})}$$

$$= \frac{1/4 + 1/8}{1/4 + 1/8 + 1/4} = \frac{3/8}{5/8} = 3/5$$

(d) $P(X^2 = 4) = P(X \in \{-2,2\}) = 1/4 + 1/4 = 1/2$

(e) $(-2)^2 = 4, (-1)^2 = 1, 0^2 = 0, 1^2 = 1, 2^2 = 4 \Rightarrow \text{range}(Y) = \{0,1,4\}$

(f) $P(Y=0) = P(X=0) = 1/4$

$P(Y=1) = P(X \in \{-1,1\}) = 1/4$

$P(Y=2) = P(X \in \{-2,2\}) = 1/2$

$$p_Y(y_k) = \begin{cases} 1/4, & y_k = 0, 1 \\ 1/2, & y_k = 4 \\ 0, & \text{else} \end{cases}$$

Problem 4

\underline{Problem 4}\\

Let random variable X be the number of trials you need to open the door, and let K_i be the event that the i^{it} key selected opens the door. \\

(a) We have\par

$p_x(1)=P(K_1)=\frac{1}{5}$ \par

$p_x(2)=P(\bar{K_1})P(K_2|\bar{K_1})=\frac{4}{5}\frac{1}{4}=\frac{1}{5}$ \par

$p_x(3)=P(\bar{K_1})P(\bar{K_2}|\bar{K_1})P(K_3|\bar{K_1}\cap \bar{K_2})=\frac{4}{5}\frac{3}{4}\frac{1}{3}=\frac{1}{5}$\\

Proceeding similarly, we see that the PMF of X is
$\rightarrow$ \fbox{$p_x(x)=\frac{1}{5}$} for $x=1,2,3,4,5.$ \\

(b) We have \par

$p_x(1)=P(K_1)=\frac{2}{10}$ \par

$p_x(2)=P(\bar{K_1})P(K_2|\bar{K_1})=\frac{8}{10}\frac{2}{9}$ \par

$p_x(3)=P(\bar{K_1})P(\bar{K_2}|\bar{K_1})P(K_3|\bar{K_1}\cap\bar{K_2})=\frac{8}{10}\frac{7}{9}\frac{2}{8}$ \\

Proceeding similarly, we see that the PMF of X is
$\rightarrow$ \fbox{$p_x(x)=\frac{2(10-x)}{90}$} for $x=1,2,3,...,10.$ \\

Problem 5

2) (a)

$$p_X(x_k) = \begin{cases} 1/10, & x_k = 0 \\ 1/4, & x_k = 1 \\ 1/5, & x_k = 2 \\ 1/10, & x_k = 3 \\ 1/5, & x_k = 4 \\ 3/20, & x_k = 5 \\ 0, & \text{else} \end{cases}$$

use jumps in $F_X(x)$.

(b)

$$P(X \geq 3) = p_X(3) + p_X(4) + p_X(5) = 1/10 + 1/5 + 3/20 = 9/20$$

$$= 1 - F_X(3^-)$$

(c)

$$P(X=5 \mid X \geq 3) = \frac{P(\{X=5\} \cap \{X \geq 3\})}{P(X \geq 3)}$$

$$= \frac{P(X=5)}{P(X \geq 3)} = \frac{3/20}{9/20} = 1/3$$

(d)

If I set the threshold at 1, I get 90% pass
" " " " " " 2, I get 65% pass
" " " " " " 3, I get 45% pass

$\Rightarrow$ set threshold at $X=2$.