Problem 1

a. The sample space of the experiment of rolling 2 five-sided dice is $\Omega = \{1,2,3,4,5\} \times \{1,2,3,4,5\} = \{1,2,3,4,5\}^2$. Overall, there are 25 possible outcomes, which are all equiprobable, since the 2 dice are fair and the rolls are independent of each other. It is useful to define the following events

$A_1 =$ "at least one of the dice has a 5 showing"

$A_2 =$ "at least one of the dice has a 1 showing"

i. It is

$$\mathrm{P}(A) = p(\{5,5\}) = \frac{1}{25}.$$

$$\mathrm{P}(A_1) = \sum_{j=1}^{5} p(\{5,j\}) + \sum_{i=1}^{4} p(\{i,5\})$$

$$= \frac{5}{25} + \frac{4}{25} = \frac{9}{25}.$$

Moreover,

$$\mathrm{P}(A \cap A_1) = p(\{5,5\}) = \frac{1}{25}.$$

We observe that

$$\mathrm{P}(A \cap A_1) = \frac{1}{25} \neq \frac{9}{25^2} = \mathrm{P}(A)\mathrm{P}(A_1).$$

Thus, the events A and A_1 are not independent.

ii. It is

$$\mathrm{P}(A_2) = \sum_{j=1}^{5} p(\{1,j\}) + \sum_{i=2}^{5} p(\{i,1\}) = \frac{9}{25}.$$

Moreover,

$$\mathrm{P}(A \cap A_2) = p(\{\varnothing\}) = 0 \neq \frac{9}{25^2} = \mathrm{P}(A)\mathrm{P}(A_2).$$

So, the events A and A_2 are not independent.

b. Let's define the events

$B_1 =$ "I get doubles"

$B_2 =$ "at least one of the dice has a 3 showing"

i. It is

$$\mathrm{P}(B) = p(\{4,4\}) + p(\{3,5\}) + p(\{5,3\}) = \frac{3}{25}.$$

$$\mathrm{P}(B_1) = \sum_{i=1}^{5} p(\{i,i\}) = \frac{5}{25}.$$

Moreover,

$$\mathrm{P}(B \cap B_1) = p(\{4,4\}) = \frac{1}{25} \neq \frac{3 \cdot 5}{25^2} = \mathrm{P}(B)\mathrm{P}(B_1).$$

Thus, the events B and B_1 are not independent.

ii. It is

$$\mathrm{P}(B \cap B_2) = p(\{3,5\}) + p(\{5,3\}) = \frac{2}{25}.$$

Thus, we have

$$\mathrm{P}(B_2|B) = \frac{P(B \cap B_2)}{P(B)} = \frac{2/25}{3/25} = \frac{2}{3}.$$

iii. It is

$$\mathrm{P}(B \cap A_1) = p(\{3,5\}) + p(\{5,3\}) = \frac{2}{25}.$$

So,

$$\mathrm{P}(A_1|B) = \frac{\mathrm{P}(B \cap A_1)}{P(B)} = \frac{2/25}{3/25} = \frac{2}{3}.$$

Problem 2

On m, any pair: 4 choices

T, any pair but m: 3 choices

W, any pair but T: 3 choices

R, any pair but W: 3 choices

F, any pair but R: 3 choices

$4 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 324$ possibilities

(b) (a) A: event of no complete pair

$P(A) = \frac{|A|}{|S|} = \frac{|A|}{\binom{16}{4}}$

The four shoes in an outcome in A must be from distinct pairs. There are $\binom{8}{4}$ ways to choose these 4 pairs. For each pair, there are $\binom{2}{1}$ to select which shoe (left or right) is in the outcome for each pair.

$$\Rightarrow |A| = \binom{8}{4}\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1} = 1120$$

$$\Rightarrow P(A) = \frac{1120}{1820} = \frac{8}{13} = 0.6154$$

(ii) B: event of <u>two</u> <u>complete pairs</u>

$|B| = \binom{8}{2} = 28$

choose the two pairs

$\Rightarrow P(B) = \frac{|B|}{|S|} = \frac{28}{1820} = 0.01538$

C: event of <u>exactly</u> <u>one</u> <u>complete</u> <u>pair</u>

$P(C) = 1 - P[A] - P[B]$

$= 0.3692$

Problem 3

4) (a) A: event fish are all of same kind

$$P[A] = \frac{|A|}{|S|} = \frac{\overbrace{\binom{r}{1}}^{\text{kinds of fish}} \overbrace{\binom{n}{5}}^{\text{choose 5 fish from n}}}{\underbrace{\binom{nr}{5}}_{\text{choose 5 fish from nr}}}$$

(b) A: event are each of a different kind

$$P[A] = \frac{|A|}{|S|} = \frac{\overbrace{\binom{r}{5}}^{\text{choose five kinds}} \overbrace{\left[\binom{n}{1}\right]^5}^{\text{choose 1 fish from each of 5 kinds}}}{\binom{nr}{5}}$$

(c) A: event 3 of one kind, 2 are 2 different kinds

$$P[A] = \frac{|A|}{|S|} = \frac{\overbrace{\binom{r}{1}}^{\text{choose kind for 3 fish}} \overbrace{\binom{r-1}{2}}^{\text{choose other two kinds}} \overbrace{\binom{n}{3}}^{\text{choose fish of main kind}} \overbrace{\left[\binom{n}{1}\right]^2}^{\text{choose fish of other two kinds}}}{\binom{nr}{5}}$$

Problem 4

Let A: 3 defective parts out of 5, T_2: shipment from the second truck

(a) $P(A) = P(T_1)P(A|T_1) + P(T_2)P(A|T_2) + P(T_3)P(A|T_3) = \frac{\binom{5}{3}\binom{15}{2}}{\binom{20}{5}} \cdot \frac{1}{3} + \frac{\binom{10}{3}\binom{30}{2}}{\binom{40}{5}} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}$

(b) $P(T_2|A) = \frac{P(A|T_2)P(T_2)}{P(A)} = \frac{\frac{\binom{10}{3}\binom{30}{2}}{\binom{40}{5}} \cdot \frac{1}{3}}{P(A)}$

Problem 5

5) (a)

$$P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$
$$= 0.05 + 0.10 + 0.15 + 0.20$$
$$= 0.50$$

(b)
$$P(X \le 3 \mid X \ge 2) = \frac{P(\{X \ge 2\} \cap \{X \le 3\})}{P(\{X \ge 2\})}$$
$$= \frac{P(2 \le X \le 3)}{P(X \ge 2)}$$
$$= \frac{P(X=2) + P(X=3)}{P(X=2) + P(X=3) + P(X=4) + P(X=5)}$$
$$= \frac{0.35}{0.85} = 7/17$$

(c)
$$P(\text{bad chip} \le 3) = P(\text{bad chips} \le 3 \mid X)\,P(X) + P(\text{bad chips} \le 3 \mid Y)\,P(Y)$$
$$= 0.50 \cdot 0.25 + 0.85 \cdot 0.75$$
$$= 1/8 + 17/20 \cdot 3/4$$
$$= 61/80$$

(d)

Y. Probabilities of small number of failures are large compared to X.