Homework 1 Solutions

Problem 1

1) (a) $H =$ "head"

$T =$ "tail"

Note: order matters, so these are distinct

$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

(b)

$\Omega = \{0, 1, 2, 3\}$

(c) $A =$ "apple"

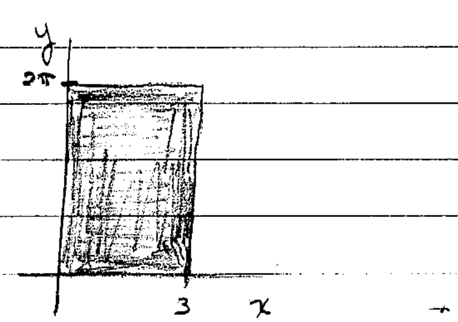
$B =$ "banana"

$O =$ "orange"

$\Omega = \{\{A, B\}, \{A, O\}, \{B, B\}, \{B, O\}, \{O, O\}\}$

(d) $\Omega = [0, 3]$ "the interval from 0 to 3"

(e) $\Omega = \{(x, y) : 0 \le x \le 3, \ 0 \le y \le 2\pi\}$



Problem 2

(a)

English:

"A_1 or (A_2 and (A_3 or A_4))"

sets:

$$A_1 \cup (A_2 \cap (A_3 \cup A_4))$$

(b)

$$P(A_1 \cup (A_2 \cap (A_3 \cup A_4)))$$

$$= P(A_1) + P(A_2 \cap (A_3 \cup A_4)) - P(A_1 \cap (A_2 \cap (A_3 \cup A_4)))$$

$$= P(A_1) + P((A_2 \cap A_3) \cup (A_2 \cap A_4)) - P((A_1 \cap A_2 \cap A_3) \cup (A_1 \cap A_2 \cap A_4))$$

note: $A_2 \cap A_2 = A_2$

$$= P(A_1) + P(A_2 \cap A_3) + P(A_2 \cap A_4) - P(A_2 \cap A_3 \cap \cancel{A_2} \cap A_4)$$

$$- (P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4) - P(\cancel{A_1 \cap A_2} \cap A_3 \cap A_1 \cap A_2 \cap A_4))$$

$$\overset{\text{indep}}{=} P(A_1) + P(A_2)P(A_3) + P(A_2)P(A_4) - P(A_2)P(A_3)P(A_4)$$

$$- P(A_1) \cdot P(A_2) \cdot P(A_3) - P(A_1)P(A_2)P(A_4) + P(A_1)P(A_2)P(A_3)P(A_4)$$

$$= 1/2 + 1/2 \cdot 1/2 + 1/2 \cdot 1/2 - 1/2 \cdot 1/2 \cdot 1/2$$

$$- 1/2 \cdot 1/2 \cdot 1/2 - 1/2 \cdot 1/2 \cdot 1/2 + 1/2 \cdot 1/2 \cdot 1/2 \cdot 1/2$$

$$= 1/2 + 1/4 + 1/4 - 1/8 - 1/8 - 1/8 + 1/16$$

$$= 11/16$$

Problem 3

3)

(a)

$P(A \cap B) = P(\emptyset) = 0$

A, B disjoint (mutually exclusive)

(b) $P(A \cup B) = P(A) + P(B)$

A, B disjoint

$\Rightarrow P(B) = 1/4$

(c)

$P(A \cap \bar{B}) = P(A) = 3/8$

A, B disjoint

(d) $P(A \cup \bar{B}) = P(A) + P(\bar{B}) - P(A \cap \bar{B})$

$= P(\bar{B})$

$= 1 - P(B)$

$= 3/4$

(e) $P(A \cap B) = 0 \neq 3/8 \cdot 1/4$ NO

(f) $P(C \cap D) = P(C)P(D)$

C, D indep

$\Rightarrow P(D) = 2/3$

(g) $P(C \cup D) = P(C) + P(D) - P(C \cap D)$

$= 1/2 + 2/3 - 1/3 = 5/6$

(h) $P(C \mid D) = P(C) = 1/2$

C, D indep

(i) $P(C \cap \bar{D}) \overset{C, D \text{ indep}}{=} P(C)P(\bar{D})$

$= P(C)(1 - P(D))$

$= 1/2 (1 - 2/3) = 1/6$

(j) $P(C \cup \bar{D}) = P(C) + P(\bar{D}) - P(C \cap \bar{D})$

$= 1/2 + (1 - 2/3) - 1/6 = 2/3$

(k) $P(\bar{C} \cap \bar{D}) \overset{C, D \text{ indep}}{=} P(\bar{C})P(\bar{D})$

$= (1 - 1/2)(1 - 2/3) = 1/6$

(l) Yes, per lecture:

C, D indep $\Rightarrow$ C, $\bar{D}$ indep.

(already used in (i) above)

$P(C \cap \bar{D}) = P(C) - P(C \cap D)$

$= P(C) - P(C)P(D)$

$= P(C)(1 - P(D))$

$= P(C)P(\bar{D})$

Problem 4

4) (a)

$$P(D_3) = P(D_3 \mid D_2)\,P(D_2) + P(D_3 \mid \bar{D}_2)\,P(\bar{D}_2)$$

$$= 0.9 \cdot 0.8 + 0.5 \cdot 0.2$$

$$= 0.82$$

(b) $$P(D_2 \mid D_3) = \frac{P(D_3 \mid D_2)\,P(D_2)}{P(D_3)} = \frac{0.9 \cdot 0.8}{0.82} = 0.72/0.82$$

(c) $$P(D_1 \cap D_2 \cap D_3) = P(D_1)\,P(D_2 \cap D_3)$$

$$= P(D_1)\,P(D_2)\,P(D_3 \mid D_2)$$

$$= 0.8 \cdot 0.8 \cdot 0.9$$

$$= 0.576$$

(d) $$P(\{D_1 \cap D_2 \cap \bar{D}_3) \cup (D_1 \cap \bar{D}_2 \cap D_3) \cup (\bar{D}_1 \cap D_2 \cap D_3))$$

$$= P(D_1 \cap D_2 \cap \bar{D}_3) + P(D_1 \cap \bar{D}_2 \cap D_3) + P(\bar{D}_1 \cap D_2 \cap D_3)$$

$$= P(D_1)\,P(D_2)\,P(\bar{D}_3 \mid D_2) + P(D_1)\,P(\bar{D}_2)\,P(D_3 \mid \bar{D}_2)$$

$$+ P(\bar{D}_1)\,P(D_2)\,P(D_3 \mid D_2)$$

$$= 0.8 \cdot 0.8 \cdot 0.1 + 0.8 \cdot 0.2 \cdot 0.5 + 0.2 \cdot 0.8 \cdot 0.9$$

$$= 0.064 + 0.08 + 0.144$$

$$= 0.288$$

Problem 5

We let A_i denote the event that we win in game i. Then,

$$\mathrm{P}(A_1) = 1 - \mathrm{P}(A_1^c) = 1 - \left(\frac{5}{6}\right)^4 = 0.518,$$

and

$$\mathrm{P}(A_2) = 1 - \mathrm{P}(A_2^c) = 1 - \left(\frac{35}{36}\right)^{24} = 0.491.$$

Since $\mathrm{P}(A_1) > \mathrm{P}(A_2)$, we would play game 1.