HW 10 Solutions

ECE 314 Introduction to Probability and Random Processes Spring 2013
May 1, 2013

Problem 1

First we need to find $\Phi(\frac{2-\alpha}{2})$.

Since we want 95% confidence, $1 - \alpha = .95$, so $\alpha = .05$.

$\frac{2-\alpha}{2} = .975$. Using z-function table, $\Phi(.975) = 1.96$.

Now you can solve for $N$.

$$\frac{\sqrt{p - p^2}}{\sqrt{N}} \Phi \left( \frac{2 - \alpha}{2} \right) \leq .01$$

$$\frac{\sqrt{p - p^2}}{\sqrt{N}} \cdot 1.96 \leq .01$$

$$\frac{\sqrt{p - p^2}}{.01} \cdot 1.96 \leq \sqrt{N}$$

$$\frac{(p - p^2)}{(.01)^2} \cdot (1.96)^2 \leq N$$

For $P(A) = .5$, $N=9604$.
For $P(A) = .01$, $N=381$. Remember, samples are discrete values.

Problem 2

a)

The good, unbiased estimate is the average of the sample set. The average of the sample space is 1.52.
b)

\[
\sigma^2_{\text{estimator}} = \text{Var} \left( \frac{1}{N} \sum_{i=1}^{10} X_i \right)
\]

\[
\sigma^2_{\text{estimator}} = \frac{N}{N^2} \text{Var}(X_i)
\]

\[
= \frac{\sigma^2}{N} = \frac{4}{10}
\]

c)

You want \( P(a < \mu < b) = .95 \)

Use \( 1 - \alpha = .95 \Rightarrow \alpha = .05. \)

So \( \frac{2-\alpha}{2} = .975 \) and \( \Phi(.975) = 1.96. \)

\[
\left( -1.96 < \frac{\hat{\mu} - \mu}{\sigma/\sqrt{N}} < 1.96 \right)
\]

\[
\left( -1.96 \cdot \frac{\sigma}{\sqrt{N}} < \hat{\mu} - \mu < 1.96 \cdot \frac{\sigma}{\sqrt{N}} \right)
\]

\[
\left( \hat{\mu} - 1.96 \cdot \frac{\sigma}{\sqrt{N}} < \mu < \hat{\mu} + 1.96 \cdot \frac{\sigma}{\sqrt{N}} \right)
\]

Plug in values: \( N = 10, \sigma = 2, \hat{\mu} = 1.52 \Rightarrow [a, b] = [.2804, 2.7596] \)

d)

You want to decrease the length of the interval by half.

\[
-\frac{\sigma}{\sqrt{N}} \Phi(.975) = \frac{.2804 - \hat{\mu}}{2}
\]

\[
-\frac{1}{\sqrt{N}} = \frac{.2804 - 1.52}{2\sigma} \cdot \frac{1}{1.96}
\]

\[
\sqrt{N} = \frac{-2\sigma}{.2804 - 1.52} \cdot 1.96
\]

\[
N = \frac{4\sigma^2}{(.2804 - 1.52)^2 \cdot (1.96)^2}
\]
N= 40. You can check this is correct by plugging \(N = 40\) into part c. The original length was: \(2.7596 - 0.2804 = 2.4792\). Using \(N = 40\), the new length is: \(2.1398 - 0.9002 = 1.2396\) which is about half of the original length.

**Problem 3**

a) 
\[\alpha = 0.01 \rightarrow \Phi\left(\frac{2-\alpha}{2}\right) = \Phi(0.995) = 2.57\]
Using similar steps as in problem 2c:
\[a = \hat{\mu} - \frac{2.57\sigma}{\sqrt{N}} = 4.2430\]
\[b = \hat{\mu} + \frac{2.57\sigma}{\sqrt{N}} = 4.7570\]

b) 
Find the minimum number of samples such that \(|\mu - \frac{1}{N} \sum_{i=1}^{N} X_i| < 0.05\)
95% confidence, \(\Phi(0.975) = 1.96\).

\[|\mu - \hat{\mu}| = -\frac{\sigma}{\sqrt{N}} \Phi(0.975) < 0.05\]
\[N > \frac{\sigma \cdot 1.96^2}{0.05^2}\]
\[N = 1537\]

99% confidence, \(\Phi(0.95) = 2.576\)

\[|\mu - \hat{\mu}| = -\frac{\sigma}{\sqrt{N}} \Phi(0.975) < 0.05\]
\[N > \frac{\sigma \cdot 2.576^2}{0.05^2}\]
\[N = 2655\]

Note, you may get \(N = 2642\) if you used 2.57 instead of 2.576. This depends on the detail of the function table you are using.