HW 10 Solutions

ECE 314 Introduction to Probability and Random Processes Spring 2013

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Problem 1

First we need to find $\Phi(\frac{2-\alpha}{2})$.
Since we want 95% confidence, $1 - \alpha = .95$, so $\alpha = .05$.
$\frac{2-\alpha}{2} = .975$. Using z-function table, $\Phi(.975) = 1.96$.
Now you can solve for N.

$$\frac{\sqrt{p-p^2}}{\sqrt{N}}\Phi\left(\frac{2-\alpha}{2}\right) \leq .01$$
$$\frac{\sqrt{p-p^2}}{\sqrt{N}} \cdot 1.96 \leq .01$$
$$\frac{\sqrt{p-p^2}}{.01} \cdot 1.96 \leq \sqrt{N}$$
$$\frac{(p-p^2)}{(.01)^2} \cdot (1.96)^2 \leq N$$

For $P(A) = .5$, N=9604.
For $P(A) = .01$, N=381. Remember, samples are discrete values.

Problem 2

a)

The good, unbiased estimate is the average of the sample set. The average of the sample space is 1.52.

b)

$$\begin{aligned}
\sigma^2_{estimator} &= Var\left(\frac{1}{N}\sum_{i=1}^{10} X_i\right) \\
\sigma^2_{estimator} &= \frac{N}{N^2} Var(X_i) \\
&= \frac{\sigma^2}{N} = \frac{4}{10}
\end{aligned}$$

c)

You want $P(a < \mu < b) = .95$
Use $1 - \alpha = .95 \Rightarrow \alpha = .05$.
So $\frac{2-\alpha}{2} = .975$ and $\Phi(.975) = 1.96$.

$$\begin{aligned}
&\left(-1.96 < \frac{\hat{\mu} - \mu}{\sigma/\sqrt{N}} < 1.96\right) \\
&\left(-1.96 \cdot \frac{\sigma}{\sqrt{N}} < \hat{\mu} - \mu < 1.96 \cdot \frac{\sigma}{\sqrt{N}}\right) \\
&\left(\hat{\mu} - 1.96 \cdot \frac{\sigma}{\sqrt{N}} < \mu < \hat{\mu} + 1.96 \cdot \frac{\sigma}{\sqrt{N}}\right)
\end{aligned}$$

Plug in values: $N = 10, \sigma = 2, \hat{\mu} = 1.52 \Rightarrow [a, b] = [.2804, 2.7596]$

d)

You want to decrease the length of the interval by half.

$$\begin{aligned}
-\frac{\sigma}{\sqrt{N}}\Phi(.975) &= \frac{.2804 - \hat{\mu}}{2} \\
-\frac{1}{\sqrt{N}} &= \frac{.2804 - 1.52}{2\sigma} \cdot \frac{1}{1.96} \\
\sqrt{N} &= \frac{-2\sigma}{.2804 - 1.52} \cdot 1.96 \\
N &= \frac{4\sigma^2}{(.2804 - 1.52)^2} \cdot (1.96)^2
\end{aligned}$$

N= 40. You can check this is correct by plugging $N = 40$ into part c.
The original length was: 2.7596 - .2804 = 2.4792
Using $N = 40$, the new length is: 2.1398 - .9002 = 1.2396 which is about half of the original length.

Problem 3

a)

$\alpha = .01 \rightarrow \Phi(\frac{2-\alpha}{2}) = \Phi(.995) = 2.57$
Using similar steps as in problem 2c:

$$a = \hat{\mu} - \frac{2.57\sigma}{\sqrt{N}} = 4.2430 \qquad\qquad b = \hat{\mu} + \frac{2.57\sigma}{\sqrt{N}} = 4.7570$$

b)

Find the minimum number of samples such that $|\mu - \frac{1}{N}\sum_{i=1}^{N} X_i| < 0.05$
95% confidence, $\Phi(.975) = 1.96$.

$$|\mu - \hat{\mu}| = -\frac{\sigma}{\sqrt{N}}\Phi(.975) < .05$$
$$N > \frac{\sigma \cdot 1.96^2}{.05^2}$$
$$N = 1537$$

99% confidence, $\Phi(.995) = 2.576$

$$|\mu - \hat{\mu}| = -\frac{\sigma}{\sqrt{N}}\Phi(.975) < .05$$
$$N > \frac{\sigma \cdot 2.576^2}{.05^2}$$
$$N = 2655$$

Note, you may get $N = 2642$ if you used 2.57 instead of 2.576. This depends on the detail of the function table you are using.