HW 7 Solutions

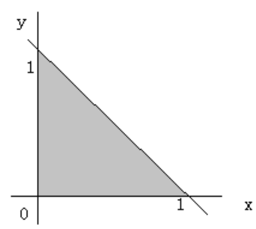
ECE 314 Introduction to Probability and Random Processes Spring 2013

April 9, 2013

Problem 1

(a)

Region where $f_{X,Y}(x,y)$is non-zero.



$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

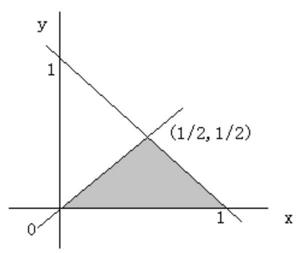
$$= \begin{cases} \int_0^{1-x} 3(x+y)dy = 3x(1-x) + \frac{3}{2}y^2\Big|_0^{1-x} = \frac{3}{2} - \frac{3}{2}x^2, & 0 \le x \le 1 \\ 0, & else \end{cases}$$

$$f_Y(y) = \begin{cases} \int_0^{1-y} 3(x+y)dx = \frac{3}{2} - \frac{3}{2}y^2, & 0 \le y \le 1 \\ 0, & else \end{cases}$$

(b)

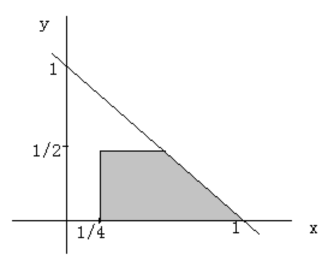
$$\begin{aligned} P(x \geq \frac{1}{2}) &= \int_{\frac{1}{2}}^{1} f_X(x)dx \\ &= \int_{\frac{1}{2}}^{1} (\frac{3}{2} - \frac{3}{2}x^2)dx \\ &= (\frac{3}{2}x - \frac{1}{2}x^3)\Big|_{\frac{1}{2}}^{1} \\ &= \frac{3}{2} - \frac{1}{2} - (\frac{3}{4} - \frac{1}{16}) \\ &= \frac{5}{16} \end{aligned}$$

(c)



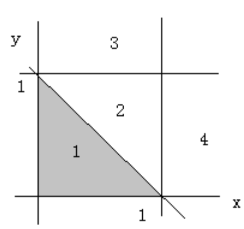
$$
\begin{aligned}
P(X \geq Y) &= \int_0^{1/2} \int_y^{1-y} 3(x+y)dxdy \\
&= \int_0^{1/2} (\frac{3}{2}x^2 + 3xy)\Big|_y^{1-y} dy \\
&= \int_0^{1/2} (\frac{3}{2}(1-y)^2 + 3(1-y)y) - (\frac{3}{2}y^2 + 3y^2)dy \\
&= \int_0^{1/2} (\frac{3}{2} - 6y^2)dy \\
&= \frac{3}{2}y - 2y^3\Big|_0^{1/2} \\
&= \frac{1}{2}
\end{aligned}
$$

(d)



$$
\begin{aligned}
P(x > 1/4|Y < 1/2) &= \frac{P((X > 1/2) \cap (Y < 1/2))}{P(Y < 1/2)} \\
&= \frac{\int_0^{1/2} \int_{1/4}^{1-y} 3(x+y)dxdy}{\int_0^{1/2} (\frac{3}{2} - \frac{3}{2}y^2)dy} \\
&= \frac{\int_0^{1/2} (\frac{3}{2}x^2 + 3xy)\Big|_{1/4}^{1-y} dy}{(\frac{3}{2} - \frac{1}{2}y^3)\Big|_0^{1/2}} \\
&= \frac{\int_0^{1/2} (3/2(1-y)^2 + 3(1-y)y) - (3/32 + 3/4y)dy}{3/4 - 1/16} \\
&= \frac{16}{11} \int_0^{1/2} (45/32 - 3/4y - 3/2y^2)dy \\
&= \frac{16}{11}(45/32y - 3/8y^2 - 1/2y^3)\Big|_0^{1/2} \\
&= \frac{35}{44}
\end{aligned}
$$

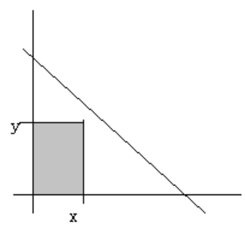
(e)



$$F_{XY}(x,y) = P(X \leq x, Y \leq y) = \int \int_D f_{XY}(x,y)dxdy$$

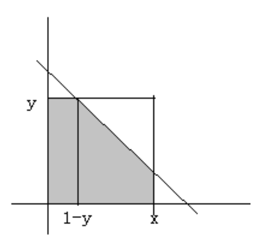
When $x < 0$ or $y < 0$, $F_{XY}(x, y) = 0$.
When $x > 1$ and $y > 1$, $F_{XY}(x, y) = 1$.
Then seperate the rest of the plain into four regions.

For region 1



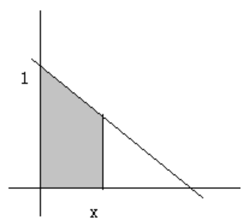
$$\begin{aligned} F_{XY}(x,y) &= \int_0^x \int_0^y 3(u+v)dvdu \\ &= \int_0^x 3(uv + 1/2v^2)\Big|_{v=0}^{v=y} du \\ &= 3(1/2u^2y + 1/2uy^2)\Big|_{u=0}^{u=x} \\ &= 3/2(x^2y + xy^2) \end{aligned}$$

For region 2



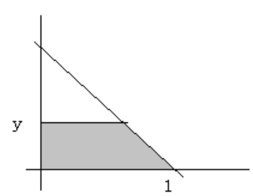
$$
\begin{aligned}
F_{XY}(x,y) &= \int_0^{1-y}\int_0^{y} 3(u+v)dvdu + \int_{1-y}^{x}\int_0^{1-u} 3(u+v)dvdu \\
&= 3/2((1-y)^2y + (1-y)y^2) + \int_{1-y}^{x} 3(uv + 1/2v^2)\Big|_{v=0}^{v=1-u} du \\
&= 3/2((1-y)^2y + (1-y)y^2) + 3/2(u - 2/3u^3)\Big|_{u=1-y}^{u=x} \\
&= 3/2((1-y)^2y + (1-y)y^2) + 3/2x - x^3 - 3/2(1-y) + (1-y)^3
\end{aligned}
$$

For region 3



$$
\begin{aligned}
F_{XY}(x,y) &= \int_0^{x}\int_0^{1-u} 3(u+v)dvdu \\
&= \int_0^{x} 3(1/2v^2 + uv)\Big|_{v=0}^{v=1-u} du \\
&= 3/2u - u^3\Big|_{u=0}^{u=x} \\
&= 3/2x - x^3
\end{aligned}
$$

For region 4



$$
\begin{aligned}
F_{XY}(x,y) &= \int_0^y \int_0^{1-v} 3(u+v)\,dudv \\
&= \int_0^y 3(1/2u^2+uv)\Big|_{u=0}^{u=1-v} dv \\
&= 3/2v - v^3\Big|_{v=0}^{v=y} \\
&= 3/2y - y^3
\end{aligned}
$$

Problem 2

(a)

$$
\begin{aligned}
E[X] &= E[cos\theta] = \int_0^{2\pi} \frac{1}{2\pi} cos\theta d\theta = \frac{sin\theta}{2\pi}\Big|_0^{2\pi} = 0 \\
E[Y] &= E[sin\theta] = \int_0^{2\pi} \frac{1}{2\pi} sin\theta d\theta = \frac{-cos\theta}{2\pi}\Big|_0^{2\pi} = 0 \\
E[XY] &= E[sin\theta cos\theta] = E[\frac{1}{2}sin2\theta] = \frac{1}{4\pi}\int_0^{2\pi} sin2\theta d\theta = \frac{-1}{8\pi}cos2\theta\Big|0^{2\pi} = 0
\end{aligned}
$$

$Cov(X,Y) = E[XY] - E[X]E[Y] = 0 \Rightarrow$ X,Y uncorrelated

Consider X=1, then Y=0 (i.e. $f_{Y|X}(y|1) = \partial(y)$)
X=0, then Y=-1 or 1 equally likely (i.e. $f_{Y|X}(y|0) = \frac{1}{2}\partial(y-1/2)+\frac{1}{2}\partial(y+1/2)$)
Since $f_{Y|X}(y|1) \neq f_{Y|X}(y|0)$, X and Y are <u>not</u> independent.

(b)

with $\rho = 0$,

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma_x\sigma_y} exp\Big[\frac{-1}{2}\Big\{\frac{(x-E[X])^2}{\sigma_x^2} + \frac{(y-E[Y])^2}{\sigma_y^2}\Big\}\Big] \\ &= \frac{1}{2\pi\sigma_x\sigma_y} exp\Big[\frac{-(x-\mu_x)^2}{2\sigma_x^2}\Big] exp\Big[\frac{-(y-\mu_y)^2}{2\sigma_y^2}\Big] \\ &= \frac{1}{\sqrt{2\pi\sigma_x^2}} exp\Big[\frac{-(x-\mu_x)^2}{2\sigma_x^2}\Big] \frac{1}{\sqrt{2\pi\sigma_y^2}} exp\Big[\frac{-(y-\mu_y)^2}{2\sigma_y^2}\Big] \\ &= f_x(x) f_y(y) \end{aligned}$$

Therefore X and Y are independent

Problem 3

(a)

We have

$$\begin{aligned} P(Y \leq y | X = x) &= P(X + Z \leq y | X = x) \\ &= P(x + Z \leq y | X = x) \\ &= P(x + Z \leq y)\ (X and Z independent) \\ &= P(Z \leq y - x) \end{aligned}$$

By differentiating both sides we obtain the desired equation.

(b)

$$g(y) = E(sgn(X)|Y = y) = \int_{-\infty}^{\infty} sgn(x) f_{X|Y}(x|y) dx$$

To find the conditional pdf of X given Y, we use

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}, where f_X(x) = \begin{cases} \frac{1}{2}, & -1 \leq X \leq 1 \\ 0, & otherwise \end{cases}$$

Since X and Z are independent,

$$f_{Y|X}(y|x) = f_Z(y-x) \Rightarrow Y|\{X=x\} \sim N(x,1)$$

To find $f_Y(y)$ we integrate $f_{Y|X}(y|x)f_X(x)$ over x:

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{Y|X}(y|x)f_X(x)dx \\
&= \int_{-1}^{1} \frac{1}{2\sqrt{2\Pi}} e^{-\frac{(y-x)^2}{2}} dx \\
&= \frac{1}{2}\int_{-1}^{1} \frac{1}{\sqrt{2\Pi}} e^{-\frac{(x-y)^2}{2}} dx \\
&= \frac{1}{2}(Q(y-1) - Q(y+1))
\end{aligned}$$

Note:$Q(x) = 1 - \Phi(x)$
Combining the above results, we get

$$\begin{aligned}
g(y) &= \int_{-\infty}^{\infty} sgn(x) f_{X|Y}(x|y)dx \\
&= \int_{-1}^{1} sgn(x) \frac{\frac{1}{2\sqrt{2\Pi}e^{-\frac{(y-x)^2}{2}}}}{f_Y(y)} dx \\
&= \frac{1}{2f_Y(y)}(-\int_{-1}^{0} \frac{1}{\sqrt{2\Pi}e^{-\frac{(x-y)^2}{2}}} dx + \int_{0}^{1} \frac{1}{\sqrt{2\Pi}e^{-\frac{(x+y)^2}{2}}} dx) \\
&= \frac{Q(y+1) - 2Q(y) + Q(y-1)}{Q(y-1) - Q(y+1)}
\end{aligned}$$

Problem 4

(a)

If X takes a value x between -1 and 1, the conditional PDF of Y is uniform between -2 and 2. If X takes a value x between 1 and 2, the conditional PDF of Y is uniform between -1 and 1.
Similarly,If Y takes a value y between -1 and 1, the conditional PDF of X is uniform between -2 and 2. If Y takes a value y between 1 and 2, the conditional PDF of X is uniform between -1 and 1.

(b)

We have $[X|Y = y] = \begin{cases} 0, & -2 \leq y \leq -1 \\ 1/2, & -1 < y \leq 1 \\ 0, & 1 \leq y \leq 2 \end{cases}$

$$E[X] = \int_{-\infty}^{\infty} E[X|Y = y] f_Y(y) dy = 3/10$$

(c)

By symmetry, we have $E[Y|X] = 0$ and $E[Y] = 0$