1 Random Variables

1.1 Random Variables

- Random Variable is a real valued variable whose value is determined by an underlying random experiment.

**Random Variables:** A random variable $X$ is a function from the sample space to the real numbers.

$$X : S \rightarrow \mathbb{R}$$

The range of a random variable $X$, shown by $\text{Range}(X)$ or $R_X$, is the set of possible values of $X$.

- Two important classes of random variables- discrete and continuous.

- A third class mixed random variables can be thought of as a mixture of discrete and continuous random variables.

1.2 Discrete Random Variables

- We define a discrete random variable as the following

  $X$ is a discrete random variable if its range is countable.

- A set $A$ is countable if either

  1. $A$ is a finite set such as $\{1, 2, 3, 4\}$, or
  2. It can be put in one-to-one correspondence with natural numbers (in this case, the set is said to be countably infinite).
1.3 Probability Mass Function (PMF)

- We can define the PMF as

Let $X$ be a discrete random variable with range $R_X = \{x_1, x_2, x_3, \ldots\}$ (finite or countably infinite). The function

$$P_X(x_k) = P(X = x_k), \text{ for } k = 1, 2, 3, \ldots,$$

is called the probability mass function (PMF) of $X$.

- We use $P_X$ as the standard notation where the subscript indicates that this is the PMF of the random variable $X$.

Properties of PMF:
- $0 \leq P_X(x) \leq 1$ for all $x$;
- $\sum_{x \in R_X} P_X(x) = 1$;
- for any set $A \subset R_X$, $P(X \in A) = \sum_{x \in A} P_X(x)$.

2 Cumulative Distribution Function (CDF)

The cumulative distribution function (CDF) of random variable $X$ is defined as

$$F_X(x) = P(X \leq x), \text{ for all } x \in \mathbb{R}.$$

For all $a \leq b$, we have

$$P(a < X \leq b) = F_X(b) - F_X(a) \quad (1)$$
3 Expectation

Expected value (=mean=average):
Let $X$ be a discrete random variable with range $R_X = \{x_1, x_2, x_3, \ldots\}$ (finite or countably infinite). The expected value of $X$, denoted by $EX$, is defined as

$$EX = \sum_{x_k \in R_X} x_k P(X = x_k) = \sum_{x_k \in R_X} x_k P_X(x_k).$$

Expectation is linear:
We have

- $E[aX + b] = aEX + b$, for all $a, b \in \mathbb{R}$;
- $E[X_1 + X_2 + \cdots + X_n] = EX_1 + EX_2 + \cdots + EX_n$, for any set of random variables $X_1, X_2, \cdots, X_n$. 

4 Problems

1. Let $X$ be a discrete random variable with the following PMF

$$P_X(x) = \begin{cases} 
\frac{1}{2} & \text{for } x = 0 \\
\frac{1}{3} & \text{for } x = 1 \\
\frac{1}{6} & \text{for } x = 2 \\
0 & \text{otherwise}
\end{cases}$$

(a) Find $R_X$, the range of the random variable $X$.
(b) Find $P(X \geq 1.5)$.
(c) Find $P(0 < X < 2)$.
(d) Find $P(X = 0 | X < 2)$
2. Let $X$ be a discrete random variable with the following PMF

$$P_X(x) = \begin{cases} 
0.2 & \text{for } x = -2 \\
0.3 & \text{for } x = -1 \\
0.2 & \text{for } x = 0 \\
0.2 & \text{for } x = 1 \\
0.1 & \text{for } x = 2 \\
0 & \text{otherwise} 
\end{cases}$$

Find and Plot the CDF of $X$, and $EX$. 