1 Counting Methods

- Counting methods can be used for discrete sample spaces with equally likely outcomes.

- For such a finite sample space $S$, the probability of an event $A$ is

$$P(A) = \frac{|A|}{|S|}$$

1.1 Multiplication Principle

- The principle is given as

Suppose that we perform $r$ experiments such that the $k$th experiment has $n_k$ possible outcomes, for $k = 1, 2, \cdots, r$. Then there are a total of $n_1 \times n_2 \times n_3 \times \cdots \times n_r$ possible outcomes for the sequence of $r$ experiments.

- Sampling: Choosing an element from a set. We draw a sample at random from a given set in which each element of set has equal chance of being chosen.

1. **With replacement**: While drawing multiple samples from a set, if we put each element back after each draw, we call it sampling with replacement. It also means *repetition allowed*.

2. **Without replacement**: While drawing multiple samples, we do not put each element back after every draw, i.e *repetitions not allowed*.

3. **Ordered**: Sampling in which ordering matters.

4. **Unordered**: Sampling in which ordering does not matter.

2 Ordered Sampling with Replacement

- We need to make $k$ draws from a set of $n$-elements in which ordering matters and there is repetition.

- Thus the total number of ways of choosing $k$ objects from a set with $n$ elements when ordering matters and repetition is allowed

$$n \times n \times \ldots \times n = n^k.$$
2.1 Ordered Sampling without Replacement: Permutations

- When ordering matters and repetitions are not allowed, the total number of ways of choosing \( k \) objects from a set with \( n \) elements is given as

\[
n \times (n - 1) \times \ldots \times (n - k + 1).
\]

- This is referred to as \( k \)-permutations of an \( n \)-element set:

\[
P_n^k = n \times (n - 1) \times \ldots \times (n - k + 1).
\]

- Note that for \( k > n \), we have \( P_n^k = 0 \) and if \( k = n \),

\[
P_n^n = n \times (n - 1) \times \ldots \times 1.
\]

- \( P_n^n \) is equal to \( n! \) and is pronounced as \( n \) factorial.

- In general

\[
The number of \( k \)-permutations of \( n \) distinguishable objects is given by
\]

\[
P_k^n = \frac{n!}{(n-k)!}, \text{ for } 0 \leq k \leq n.
\]

- Common notations for \( k \)-permutations of an \( n \)-element set include \( P_{n,k} \), \( nP_k \), etc.
3 Problem

1. If \( k \) people are at a party, what is the probability that at least two of them have the same birthday? Suppose that there are \( n = 365 \) days in a year and all days are equally likely to be the birthday of a specific person.

   **Hint:** Let \( A \) be the event that at least two people have the same birthday. First note that if \( k > n \), then \( P(A) = 1; \) so, let’s focus on the more interesting case where \( k \leq n \). The phrase “at least” suggests that it might be easier to find the probability of the complement event, \( P(A^c) \). This is the event that no two people have the same birthday ...

   **Solution:** Let \( A \) be the event that at least two people have the same birthday. First note that if \( k > n \), then \( P(A) = 1; \) so, let’s focus on the more interesting case where \( k \leq n \). The phrase “at least” suggests that it might be easier to find the probability of the complement event, \( P(A^c) \). This is the event that no two people have the same birthday, and we have

\[
P(A) = 1 - \frac{|A^c|}{|S|}.
\]

Thus, to solve the problem it suffices to find \( |A^c| \) and \( |S| \). Let’s first find \( |S| \). What is the total number of possible sequences of birthdays of \( k \) people? Well, there are \( n = 365 \) choices for the first person, \( n = 365 \) choices for the second person, ..., \( n = 365 \) choices for the \( k \)th person. Thus there are

\[
n^k
\]

possibilities. This is, in fact, an ordered sampling with replacement problem, and as we have discussed, the answer should be \( n^k \) (here we draw \( k \) samples, birthdays, from the set \( \{1, 2, ..., n = 365\} \)). Now let’s find \( |A^c| \). If no birthdays are the same, this is similar to finding \( |S| \) with the difference that repetition is not allowed, so we have

\[
|A^c| = P^n_k = n \times (n - 1) \times ... \times (n - k + 1).
\]

You can see this directly by noting that there are \( n = 365 \) choices for the first person, \( n - 1 = 364 \) choices for the second person, ..., \( n - k + 1 \) choices for the \( k \)th person. Thus the probability of \( A \) can be found as

\[
P(A) = 1 - \frac{|A^c|}{|S|}
= 1 - \frac{P^n_k}{n^k}.
\]
4 Unordered Sampling without Replacement: Combinations

- We want to make \( k \) draws from a set of \( n \)-elements in which ordering does not matter and repetition is not allowed.
- This means that we have to chose a \( k \)-element subset of \( A \), and is also called \( k \)-combination of the set \( A \).
- The number of \( k \)-element subsets of \( A \) is given by \( \binom{n}{k} \) and is read as \( n \) choose \( k \).
- The difference between \( \binom{n}{k} \) and \( P_k^n \) is in the ordering.
- For any \( k \)-element subset of \( A \), we can order the elements in \( k! \) ways. Thus
  \[
  P_k^n = \binom{n}{k} \times k!
  \]
  \[
  \binom{n}{k} = \frac{n!}{k!(n-k)!}
  \]
- If \( k > n \), then \( \binom{n}{k} = 0 \).
- In general
  
  The number of \( k \)-combinations of an \( n \)-element set is given by
  \[
  \binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for } 0 \leq k \leq n.
  \]
- \( \binom{n}{k} \) is also called the binomial coefficient; as the coefficients in the binomial theorem are given by \( \binom{n}{k} \).
- The binomial theorem states that for an integer \( n \geq 0 \), we have
  \[
  (a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.
  \]
- \( n \) choose \( k \) is also denoted as \( C_{n,k}, C(n,k), nCk \), etc.

4.1 Interpretation of \( \binom{n}{k} \)

We can interpret \( \binom{n}{k} \) as

The total number of ways to divide \( n \) distinct objects into two groups \( A \) and \( B \) such that group \( A \) consists of \( k \) objects and group \( B \) consists of \( n - k \) objects is \( \binom{n}{k} \).
5 Problem

2. I have 10 Algebra books and 20 probability books. I choose 5 books at random. What is the probability that I choose more than 2 probability books?

Solution:

\[ P(A) = \frac{|A|}{|S|} = \frac{\binom{10}{2}\binom{20}{3} + \binom{10}{1}\binom{20}{4} + \binom{10}{0}\binom{20}{5}}{\binom{30}{5}}. \]  (1)