1 Counting Methods

- Counting methods can be used for discrete sample spaces with equally likely outcomes.

- For such a finite sample space $S$, the probability of an event $A$ is

$$P(A) = \frac{|A|}{|S|}$$

1.1 Multiplication Principle

- The principle is given as

Suppose that we perform $r$ experiments such that the $k$th experiment has $n_k$ possible outcomes, for $k = 1, 2, \cdots, r$. Then there are a total of $n_1 \times n_2 \times n_3 \times \cdots \times n_r$ possible outcomes for the sequence of $r$ experiments.

- Sampling - Choosing an element from a set. We draw a sample at random from a given set in which each element of set has equal chance of being chosen.

1. With replacement - While drawing multiple samples from a set, if we put each element back after each draw, we call it sampling with replacement. It also means repetition allowed.

2. Without replacement - While drawing multiple samples, we do not put each element back after every draw, i.e repetitions not allowed.

3. Ordered - Sampling in which ordering matters.

4. Unordered - Sampling in which ordering does not matter.

2 Ordered Sampling with Replacement

- We need to make $k$ draws from a set of $n$-elements in which ordering matters and there is repetition.

- Thus the total number of ways of choosing $k$ objects from a set with $n$ elements when ordering matters and repetition is allowed

$$n \times n \times \cdots \times n = n^k.$$
2.1 Ordered Sampling without Replacement: Permutations

- When ordering matters and repetitions are not allowed, the total number of ways of choosing k objects from a set with n elements is given as

\[ n \times (n - 1) \times \ldots \times (n - k + 1). \]

- This is referred to as \( k \)-permutations of an n-element set:

\[ P_n^k = n \times (n - 1) \times \ldots \times (n - k + 1). \]

- Note that for \( k > n \), we have \( P_n^k = 0 \) and if \( k = n \),

\[ P_n^n = n \times (n - 1) \times \ldots \times 1. \]

- \( P_n^n \) is equal to \( n! \) and is pronounced as \( n \) factorial.

- In general

\[ P_k^n = \frac{n!}{(n-k)!}, \text{ for } 0 \leq k \leq n. \]

- Common notations for \( k \)-permutations of an n-element set include \( P_{n,k}, nPk \), etc.
3 Problem

1. If $k$ people are at a party, what is the probability that at least two of them have the same birthday? Suppose that there are $n = 365$ days in a year and all days are equally likely to be the birthday of a specific person.

*Hint: Let $A$ be the event that at least two people have the same birthday. First note that if $k > n$, then $P(A) = 1$; so, let’s focus on the more interesting case where $k \leq n$. The phrase “at least” suggests that it might be easier to find the probability of the complement event, $P(A^c)$. This is the event that no two people have the same birthday ...
4 Unordered Sampling without Replacement: Combinations

- We want to make $k$ draws from a set of $n$-elements in which ordering does not matter and repetition is not allowed.
- This means that we have to chose a $k$-element subset of $A$, and is also called $k$-combination of the set $A$.
- The number of $k$-element subsets of $A$ is given by $\binom{n}{k}$ and is read as $n$ choose $k$.
- The difference between $\binom{n}{k}$ and $P^n_k$ is in the ordering.
- For any $k$-element subset of $A$, we can order the elements in $k!$ ways. Thus
  \[ P^n_k = \binom{n}{k} \times k! \]
  \[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \]
- If $k > n$, then $\binom{n}{k} = 0$.
- In general
  \[ \binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for } 0 \leq k \leq n. \]
- $\binom{n}{k}$ is also called the binomial coefficient; as the coefficients in the binomial theorem are given by $\binom{n}{k}$.
- The binomial theorem states that for an integer $n \geq 0$, we have
  \[ (a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}. \]
- $n$ choose $k$ is also denoted as $C_{n,k}$, $C(n,k)$, $nCk$, etc.

4.1 Interpretation of $\binom{n}{k}$

We can interpret $\binom{n}{k}$ as

The total number of ways to divide $n$ distinct objects into two groups $A$ and $B$ such that group $A$ consists of $k$ objects and group $B$ consists of $n - k$ objects is $\binom{n}{k}$.
5 Problem

2. I have 10 Algebra books and 20 probability books. I choose 5 books at random. What is the probability that I choose more than 2 probability books?