Chapter 11

Transmission of Random Processes Through Systems

11.1 Definitions and Introductory Ideas

We frequently encounter situations where random processes are input to systems and we are interested in the outputs of these systems. Recall that a random process is a family of “time” functions, called sample functions or realizations - one corresponding to each outcome \( s \in S \). Each one of these time functions when input to the system produces a time function as output. Then, we can consider the output of the system, when the input is a random process, as the family of the output time functions, which in turn constitutes a random process as the output of the system. In other words, we have

\[ Y(t) = \mathcal{G}\{X(t)\} \quad (11.1) \]

where \( X(t) \) is the input random process, \( Y(t) \) is the output random process, and \( \mathcal{G}(\cdot) \) denotes the system, also called the transformation. See Fig. 11.1 for a representation of this system or the relationship implied by it.

It is conceivable that the system \( \mathcal{G}(\cdot) \) itself can have randomness, such as, parameters that are random variables affecting the operation of the system. In such a system, which can be called a random system, the output depends not only on the input time-function, but also on the outcome \( s \in S \) of the random experiment. However, it is rare that we have to deal with random systems; hence, in this chapter, we will study the transmission of random processes through deterministic systems, that is, those systems where the output depends only on the
input time-function and no additional randomness exists in the system itself other than the input time-function being random (i.e., being dependent on the outcome of $S$).

The general problem we try to solve is: Given the (deterministic) system $G(\cdot)$ and the statistics of the input random process $X(t)$, to determine the statistics of the output random process $Y(t)$. We will organize what we can say as solution to this problem according to various properties the systems satisfy. We consider subclasses of systems defined by certain properties which the members of the subclass must satisfy. Specifically, we consider:

1. Memoryless systems,
2. Time-invariant systems,
3. Linear systems.

Each of these classes of systems are characterized by a corresponding property and it is not implied that they are disjoint subclasses. In the sequel, we will give definitions of these subclasses and study the implications on determining the statistics of the output in terms of those of the input. Linear systems being most important in Electrical and Computer Engineering, we will expend most of the energy on studying the transformation of random processes through linear systems. The formulation will be developed first for continuous-time (CT) random processes. The formulation for discrete-time (DT) random processes going through DT linear systems will be presented in the last section of this chapter.
11.2 MEMORYLESS AND/OR TIME-INVARIANT SYSTEMS

11.2 Memoryless and/or Time-invariant Systems

A system represented by $G(\cdot)$ is a memoryless system if the output random process $Y(t)$ at $t = t_1 \in T$, i.e., the r.v. $Y(t_1)$ is given by

$$Y(t_1) = G[X(t_1), t_1].$$  \hspace{1cm} (11.2)

In other words, for each $t_1 \in T$, $Y(t_1)$ depends on $t_1$ and $X(t_1)$, not on $X(t)$ at $t \neq t_1$. Hence, the transformation is called memoryless. Note, however, that the dependence on $t_1$ implies that the system may be changing with time.

The following are a few examples of transformations that represent memoryless systems:

$$Y(t) = t X^2(t).$$
$$W(t) = X^2(t).$$
$$Z(t) = e^{X(t)}.$$

The first system is memoryless but depends on time; the latter two are memoryless and do not depend on time separately.

The $n$-dimensional CDF or PDF of the output process $Y(t)$ can be found from the $n$-dimensional CDF or PDF of the input process $X(t)$ simply as the problem of "$n$ functions of $n$ r.v.'s" where in this case each one of the $n$ functions involves only one of the input r.v.'s. Although conceptually simple, this may be a tedious computation, except for special cases such as input $X(t)$ being a white noise process or for $n \leq 2$.

A system $G(\cdot)$ is said to be time-invariant if "input $X(t)$ produces output $Y(t)$" implies "input $X(t-t_0)$ produces output $Y(t-t_0)$, for any $t_0 \in T$ and any $X(t)$". In other words, with a time-invariant system, if the input is delayed by $t_0$ then the output is delayed by $t_0$. This implicitly implies that the system itself is not varying with time. If there is an initial condition involved in the system, then in the above definition it is assumed that the delayed input produces the delayed output with the same initial condition. Note that the determination of the statistics of the output from those of the input for a time-invariant system without additional assumptions such as memorylessness or linearity can be hopelessly complicated.

Combining the two properties, we consider the class of systems that are memoryless and time-invariant, namely $Y(t) = g[X(t)]$, for each $t \in T$. The $n$-th order CDF of the output $Y(t)$
can be computed as follows: For $t_1, t_2, \ldots, t_n$ in $T$, we have

$$Y(t_1) = g[X(t_1)] , \quad Y(t_2) = g[X(t_2)] , \quad \ldots , \quad Y(t_n) = g[X(t_n)] ,$$ (11.3)

and

$$F_Y(y_1, y_2, \ldots, y_n; t_1, t_2, \ldots, t_n) \triangleq P(Y(t_1) \leq y_1, Y(t_2) \leq y_2, \ldots, Y(t_n) \leq y_n)$$ (11.4)

$$= \ P(g[X(t_1)] \leq y_1, \ldots, g[X(t_n)] \leq y_n)$$ (11.5)

$$= \ P((X(t_1), X(t_2), \ldots, X(t_n)) \in B_y)$$ (11.6)

$$= \int_{B_y} \cdots \int f_X(x_1, \ldots, x_n; t_1, \ldots, t_n) \, dx_1 \ldots dx_n$$ (11.7)

where $B_y = \{(x_1, x_2, \ldots, x_n) : g(x_1) \leq y_1, g(x_2) \leq y_2, \ldots, g(x_n) \leq y_n \}$. This formulation is conceptually simple, but again it may be difficult to actually implement for any non-trivial memoryless, time-invariant system, especially for $n > 2$. In any event, we have a procedure to find the $n-th$ order CDF or PDF of the output from that of the input.

Stationarity

Given a memoryless, time-invariant system, the following relationships on the stationarity of the input and output random processes hold:

- If input random process $X(t)$ is strictly stationary (or stationary of order $k$), then the output process $Y(t)$ is also strictly stationary (or stationary of order $k$).

- If input $X(t)$ is wide-sense stationary, then the output $Y(t)$ is not necessarily stationary in any sense.

It is fairly straightforward to prove these facts, although proofs will be omitted here.

Remark

As alluded to earlier, in determining the response of the system, the response (output) depends not only on the input but also on the initial state. In many instances, initial state is not specified. If nothing is said explicitly, then it is assumed and/or implied that it is zero, which is described as the system being at zero-state or at rest. If no initial time is specified, it is assumed/implied to be $t_0 = -\infty$. 
11.3 Linear Systems

Before we discuss the transmission of random processes through linear systems, we will study the basic properties of linear systems and the ways they are characterized. A linear system is defined as a system that satisfies the superposition principle, which is: If the system at initial state $A_1$ with input $x_1(t)$ produces output $y_1(t)$ and at initial state $A_2$ with input $x_2(t)$ produces output $y_2(t)$, then at initial state $c_1A_1 + c_2A_2$ with input $c_1x_1(t) + c_2x_2(t)$ it produces output $c_1y_1(t) + c_2y_2(t)$, for any constants $c_1$ and $c_2$, any initial states $A_1$ and $A_2$, and inputs $x_1(t)$ and $x_2(t)$.

An incomplete definition of the superposition principle and of linear systems is sometimes stated without any reference to the initial state, as follows: If input $x_1(t)$ produces output $y_1(t)$ and $x_2(t)$ produces $y_2(t)$, then $c_1x_1(t) + c_2x_2(t)$ produces $c_1y_1(t) + c_2y_2(t)$, for any $c_1$, $c_2$, $x_1(t)$, and $x_2(t)$. The complete definition reduces to this one if the initial conditions $A_1$ and $A_2$ are assumed 0, which may or may not be true.

In general, the response of a linear system $y(t)$ corresponding to initial state $A$ and input $x(t)$ is the sum of two components:

\[ y(t) = y_{ss}(t) + y_{zi}(t), \quad (11.8) \]

where $y_{ss}(t)$ is called the zero-state response and $y_{zi}(t)$ is called the zero-input response. The zero-state response $y_{ss}(t)$ is the response of the system to 0 initial state and $x(t)$ input. And, the zero-input response is the response of the system to initial state $A$ and input 0. The sum of the two initial conditions is $0 + A = A$; and the sum of the two inputs is $x(t) + 0 = x(t)$. Hence, by superposition principle then the initial condition $A$ and input $x(t)$ produce the sum of the two responses as implied by the above decomposition of $y(t)$.

Note that linear systems are often described/characterized by linear differential equations (DE). The solution of the DE corresponding to an initial condition and input (forcing function) corresponds to the output (response) of the system represented by the DE. Thus, techniques for solving DE in the time domain or by using Laplace transforms can be used to obtain the output of linear systems. We assume that the reader is somewhat familiar with these techniques and not study them here. For example, taking the Laplace transform of the DE and assuming zero
initial conditions, yields an algebraic equation of the form:

\[ Y(s) Q(s) = X(s) P(s) , \]  
(11.9)

where \( X(s) \) and \( Y(s) \) are the Laplace transforms of \( x(t) \) and \( y(t) \), respectively, and \( P(s) \) and \( Q(s) \) are polynomials in \( s \). This algebraic equation involving the Laplace transforms can be rearranged to yield:

\[ H(s) \triangleq \frac{P(s)}{Q(s)} = \frac{Y(s)}{X(s)} , \]  
(11.10)

where \( H(s) \) is called the transfer function of the linear system. Thus, given or having obtained \( H(s) \) and given \( X(s) \), we can find \( Y(s) \) and taking its inverse Laplace transform, we can obtain \( y(t) \), the (zero-state) output of the system.

An alternative characterization of a linear system in the time-domain is obtained in terms of the so-called impulse response of the system. The impulse response of a linear system, denoted by \( h(t, \tau) \), is defined as "the response of the system at time \( t \) to an impulse applied at \( \tau \)" with the assumption that the initial condition is zero. In other words, input \( \delta(t - \tau) \) produces the output \( h(t, \tau) \), with zero initial conditions. Now, we also know that, using the so-called sifting property of the impulse (\( \delta \) function), any input \( x(t) \) can be represented as:

\[ x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) \, d\tau . \]  
(11.11)

Since the integral can be interpreted as a continuous summation, we see that any \( x(t) \) can be represented as a continuous sum of \( \delta \)-functions at \( \tau \) with amplitude \( x(\tau) \). Now, we can invoke the superposition principle to say that if \( \delta(t - \tau) \) produces \( h(t, \tau) \), then \( x(t) \) represented as the continuous sum of \( \delta(t - \tau) \)'s with amplitude \( x(\tau) \) produces

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t, \tau) \, d\tau . \]  
(11.12)

Thus, we have a time domain expression for the output \( y(t) \) in terms of the input \( x(t) \) and the impulse response \( h(t, \tau) \).

If the linear system is time-invariant, that corresponds to the impulse response \( h(t, \tau) \) satisfying \( h(t, \tau) = h(t - \tau) \). This means for a linear time-invariant (LTI) system, the response at \( t \) due to an impulse at \( \tau \), does not depend on \( t \) and \( \tau \) separately, but depends only on \( (t - \tau) \), the difference of the two time variables, that is, the time elapsed since the \( \delta \) was applied. When this is true, we can represent the impulse response \( h(t - \tau) \) in terms of a single time variable as
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$h(\tau)$, where $\tau$ now represents the time elapsed between the time the $\delta$ is applied and the time the output is observed. For a LTI system the above expression for the output becomes

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau,$$

which we immediately recognize as the convolution integral between the input $x(\tau)$ and the impulse response $h(\tau)$. Upon a simple change of variable, the convolution integral above can also be written as

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau) h(\tau) \, d\tau.$$

Thus, the convolution operation, denoted by $y(t) = x(t) \ast h(t)$, is symmetrical in the two components. We note again that the convolution between the input and the impulse response of a LTI system yields the (zero-state) response of the system.

A system is said to be causal or physically realizable if its output at time $t$ depends on inputs at or prior to $t$, not after $t$. A system for which this is not true is called non-causal or physically unrealizable or anticipative, because it would be putting out a response due to an input before the input is applied, raising the question "how does it know that the input will be applied?" For a LTI system, the property can be expressed in terms of the impulse response of the system: A LTI system is causal if and only if its impulse response satisfies:

$$h(\tau) = 0, \quad \text{for } \tau < 0.$$  

(11.15)

So, for a causal LTI system the convolution integral for the output can be written as:

$$y(t) = \int_{-\infty}^{t} x(\tau) h(t-\tau) \, d\tau$$

(11.16)

$$= \int_{0}^{\infty} x(t - \tau) h(\tau) \, d\tau.$$  

(11.17)

Of course, if the input $x(t)$ is zero on certain parts of interval of integration, then that would also change the limits of the integration. For example, if the input is zero until $t = 0$ and $x(t)$ for $t \geq 0$, then the expression for the output would be

$$y(t) = \int_{0}^{t} x(\tau) h(t-\tau) \, d\tau$$

(11.18)

$$= \int_{0}^{t} x(t - \tau) h(\tau) \, d\tau,$$

(11.19)

for $t \geq 0$; and, $y(t) = 0$, for $t < 0$. 


Finally, we note that the Laplace transform of \( h(t) \), the impulse response of a LTI system, defined as
\[
H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt ,
\]
is called the transfer function of the system, which was discussed in the context of the Laplace transform solution of a DE representing an LTI system. Depending on \( h(t) \), the Laplace transform \( H(s) \) exists only on certain parts of the \( s \) plane, which is called the region of convergence (ROC) of \( H(s) \). ROC is where the Laplace transform integral exists. For \( H(s) \) a rational function, the associated ROC can be an infinite vertical strip or infinite vertical half-plane in the \( s \) plane. If the system is causal, then the ROC is the infinite vertical half-plane to the right of the rightmost pole of \( H(s) \). \( H(s) \) alone is not the Laplace transform of \( h(t) \); \( H(s) \) together with the associated ROC is the Laplace transform. Because a single \( H(s) \) with different ROC’s correspond to different \( h(t) \)’s. Hence, the ROC associated with an \( H(s) \) must always be specified to eliminate any ambiguity, unless the system is known to be causal, in which case the ROC is uniquely specified. \( h(t) \) and \( \{ H(s), ROC \} \) is a Laplace transform pair and satisfies all the Laplace transform properties.

The Fourier transform of \( h(t) \) defined as
\[
H(j\Omega) = \int_{-\infty}^{\infty} h(t) e^{-j\Omega t} dt ,
\]
is called the system function or complex transfer function, when it exists. The system function \( H(j\Omega) \) exists if \( s = j\Omega \), the imaginary axis in the \( s \) plane, lies in the ROC associated with the Laplace transform. \( h(t) \) and \( H(j\Omega) \) (when it exists) is a Fourier transform pair and satisfies all Fourier transform properties.

This completes a quick review of the essential properties of linear systems. For a more thorough treatment, the reader should refer to any Signals and Systems textbook used in Electrical Engineering curricula. In the next section we consider random processes going through linear systems.

11.4 Transmission of Random Processes through Linear Systems

We will now study the transmission of random processes through linear systems. We assume a certain characterization of the linear system perhaps in terms of its impulse response or
the system function is given. Also given is a characterization of the input random process. Usually the characterization of the input random process is not in terms of individual sample functions, rather it is in terms of certain statistics of the input process, such as, its mean and autocorrelation function or power spectral density or joint PDF's or CDF's. And, the objective is to determine the corresponding statistics of the output random process.

Consider a linear time-invariant (LTI) system with impulse response \( h(\tau) \). Suppose input to this system is a random process \( X(t) \) with mean \( \mu_X(t) \) and autocorrelation function (a.c.f.) \( R_{XX}(s,t) \). (Note that at this point we do not assume \( X(t) \) is w.s.s.; later we will.) Then the output of the system is a random process \( Y(t) \) expressed as

\[
Y(t) = \int_{-\infty}^{\infty} X(\tau) h(t-\tau) d\tau.
\] (11.22)

We will now determine the mean and a.c.f. of the output process. First, its mean:

\[
\mu_Y(t) = E\{Y(t)\} = E\left\{ \int_{-\infty}^{\infty} X(\tau) h(t-\tau) d\tau \right\}
\] (11.23)

\[
= \int_{-\infty}^{\infty} E\{X(\tau)\} h(t-\tau) d\tau
\] (11.24)

\[
= \int_{-\infty}^{\infty} \mu_X(\tau) h(t-\tau) d\tau
\] (11.25)

\[
= \mu_X(t) * h(t).
\] (11.26)

The second equality follows from the fact that expectation is a linear operation and since linear operations are commutative, the expected value can be moved inside the integral. Now, if the mean of the input process \( X(t) \) is constant, that is, \( \mu_X(t) = \mu_X \), which would be true if \( X(t) \) is w.s.s., then the above expression for \( \mu_Y(t) \) reduces to

\[
\mu_Y(t) = \mu_X \int_{-\infty}^{\infty} h(t-\tau) d\tau
\] (11.27)

\[
= \mu_X \int_{-\infty}^{\infty} h(\tau) d\tau
\] (11.28)

\[
= \mu_X H(j0),
\] (11.29)

which is also a constant; that is, \( \mu_Y(t) = \mu_Y \). The second equality above follows from a change of variable in the integral and the last equality follows from letting \( \Omega = 0 \) in the Fourier transform of \( h(t) \).

We can find the a.c.f. of \( Y(t) \) in two steps: First by finding the cross-correlation between \( X(t) \) and \( Y(t) \) and then finding the a.c.f. in terms of the cross-correlation:

\[
R_{YX}(s,t) = E\{Y(s) X^*(t)\}
\] (11.30)
\[ R_{YX}(s, t) = R_{XX}(s, t) \ast h(s). \] (11.34)

In the second step, we obtain \( R_{YY}(s, t) \) in terms of \( R_{YX}(s, t) \) as follows:

\[
R_{YY}(s, t) = E\{Y(s)Y^*(t)\} = E \left\{ Y(s) \int_{-\infty}^{\infty} X^*(\sigma) h^*(t - \sigma) d\sigma \right\} \]
\[ = \int_{-\infty}^{\infty} E\{Y(s)X^*(\sigma)\} h^*(t - \sigma) d\sigma \]
\[ = \int_{-\infty}^{\infty} R_{YX}(s, \sigma) h^*(t - \sigma) d\sigma \]
\[ = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} R_{XX}(\tau, \sigma) h(s - \tau) d\tau \right] h^*(t - \sigma) d\sigma. \] (11.39)

We recognize that the fourth equality is a convolution of \( h^*(\sigma) \) and \( R_{YX}(s, \sigma) \) with respect to the second variable of \( R_{YX}(s, \sigma) \), that is,

\[
R_{YY}(s, t) = R_{YX}(s, t) \ast h^*(t) = R_{XX}(s, t) \ast h(s) \ast h^*(t). \] (11.41)

Thus, we have obtained \( R_{YY}(s, t) \), the a.c.f. of the output process \( Y(t) \), in terms of the a.c.f. of the input process and the impulse response of the linear system.

Now, if we assume that the input process \( X(t) \) is w.s.s., so that, \( R_{XX}(\tau, t) = R_{XX}(\tau - t) \), then we obtain simplifications in the above expressions. The cross-correlation expression reduces to:

\[
R_{YX}(s, t) = \int_{-\infty}^{\infty} R_{XX}(\tau - t) h(s - \tau) d\tau \]
\[ = \int_{-\infty}^{\infty} R_{XX}(\sigma) h(s - t - \sigma) d\sigma. \] (11.43)

The second equality follows from a change of variable in the integral. We recognize that this expression does not depend on \( s \) and \( t \) separately, but depends on \( (s - t) \), so we can express
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\[ R_{YX}(s, t) \] as \( R_{YX}(s-t) \). We also recognize that the last expression is the convolution of \( R_{XX}(\tau) \) with \( h(\tau) \), that is,

\[ R_{YX}(\tau) = R_{XX}(\tau) \ast h(\tau). \quad (11.44) \]

Now substituting \( R_{YX}(s - \tau) \) in place of \( R_{YX}(s, \tau) \), the expression for \( R_{YY}(s, t) \), the a.c.f. of the output \( Y(t) \), in terms of \( R_{YX} \) becomes

\[ R_{YY}(s, t) = \int_{-\infty}^{\infty} R_{YX}(s - \tau) h^*(t - \tau) d\tau \quad (11.45) \]

\[ = \int_{-\infty}^{\infty} R_{YX}(s - t - \sigma) h^*(-\sigma) d\sigma. \quad (11.46) \]

We recognize that, the RHS of the second equality does not depend on \( s \) and \( t \) separately, but depends on \( s - t \). Hence, we can express \( R_{YY}(s, t) \) as \( R_{YY}(s - t) \). We also recognize that the last expression is the convolution of \( h^*(-\tau) \) with \( R_{YX}(\tau) \). Recalling that \( R_{YX}(\tau) \) is the convolution of \( h(\tau) \) and \( R_{XX}(\tau) \), we finally have

\[ R_{YY}(\tau) = R_{XX}(\tau) \ast h(\tau) \ast h^*(-\tau). \quad (11.47) \]

In obtaining the mean and a.c.f. of the output \( Y(t) \), we also obtained the result that with a LTI system, if the input process is w.s.s., then the output process is also w.s.s. For the mean and a.c.f. we obtained:

\[ \mu_Y = H(j0) \cdot \mu_X \quad (11.48) \]

\[ R_{YY}(\tau) = R_{XX}(\tau) \ast h(\tau) \ast h^*(-\tau). \quad (11.49) \]

Using the convolution property of Fourier transform pairs on the expression for \( R_{YY}(\tau) \), we obtain the relationship between the input and output power spectral densities:

\[ S_{YY}(j\Omega) = S_{XX}(j\Omega) H(j\Omega) H^*(j\Omega) \quad (11.50) \]

\[ = |H(j\Omega)|^2 S_{XX}(j\Omega). \quad (11.51) \]

Other stationarity results that hold for linear time-invariant systems are as follows:

1. If the input \( X(t) \) to a LTI system is stationary in the strict sense, then the output is also stationary in the strict sense.

2. If the input \( X(t) \) to a LTI system is stationary of order \( k \), then in general the output is not stationary in any sense (or order).
We stress the fact that the output is w.s.s. and the spectral analysis to obtain the p.s.d. of the output holds only if the input is w.s.s. If the input to the LTI system is not w.s.s. then the spectral analysis does not hold. For example, consider a LTI system and a w.s.s. process $X(t)$. If $X(t)$ is applied to the system starting at a finite time or until a finite time or over a finite time intervals, then the actual input to the system is NOT w.s.s., although $X(t)$ is w.s.s. This is because the input is not $X(t)$ over all time; it is $X(t)$ only over part of the time, hence it is not w.s.s. This would be the case when the system is represented by a linear DE with a w.s.s. $X(t)$ which is applied to the system at some initial time (which is not $-\infty$) with zero or non-zero initial conditions. The input to the system in this case is zero upto initial time and $X(t)$ after that, hence the input is NOT w.s.s. In all such cases where the input is not w.s.s., we have to obtain an expression for the output which may consist of different expressions valid at different time intervals and proceed to find its mean and a.c.f. of the output using this multi-line expression. The mean may or may not be constant and the a.c.f. $R_{YY}(s,t)$ is not a function of $(s - t)$ only. Thus, the output will not be w.s.s. in general. Below we will do two examples: first, where the input is w.s.s. and we can do the spectral analysis, and, second, where the input is not w.s.s. and therefore the output is not w.s.s.

Example 11.1 Consider a LTI system with impulse response

$$h(t) = e^{-\alpha \tau} u(\tau), \quad \alpha > 0.$$  \hspace{1cm} (11.52)

The system (complex transfer) function for this system is

$$H(j\Omega) = \frac{1}{\alpha + j\Omega}.$$  \hspace{1cm} (11.53)

Suppose a zero mean, white noise process $W(t)$ with a.c.f.

$$R_{WW}(\tau) = N_0 \delta(\tau)$$  \hspace{1cm} (11.54)

is applied as input. The power spectral density (p.s.d.) of the input is

$$S_{WW}(j\Omega) = N_0 \quad \text{for all } \Omega.$$  \hspace{1cm} (11.55)

Now we want to find the mean, a.c.f., and p.s.d. of the output $Y(t)$. We know that the system is LTI and the input is w.s.s., hence the output will also be w.s.s. and we can use the spectral analysis. First, the mean:

$$E\{Y(t)\} = E\{W(t)\} H(j0) = 0,$$  \hspace{1cm} (11.56)
Figure 11.2: The power spectral density and the autocorrelation function of the output in Example 11.1.

Since $E\{W(t)\} = 0$. Using the relation between the input and output p.s.d.'s we obtain

$$S_{YY}(j\Omega) = |H(j\Omega)|^2 S_{WW}(j\Omega) = \frac{N_0}{\alpha^2 + \Omega^2},$$

which is known as the Cauchy function. We know that this function has a known inverse Fourier transform

$$R_{YY}(\tau) = \frac{N_0}{2\alpha} e^{-\alpha |\tau|}.$$  

Thus, we find the a.c.f. of the output using its p.s.d.

Note that we can also find the a.c.f. $R_{YY}(\tau)$ through the double convolution:

$$R_{YY}(\tau) = R_{WW}(\tau) * h(\tau) * h^*(-\tau) = N_0 \delta(\tau) * e^{-\alpha \tau} u(\tau) * e^{\alpha \tau} u(-\tau).$$  

The convolution with the $\delta(\tau)$ is trivial and the convolution of the two exponentials (one right-sided, one left-sided) yields the two-sided exponential given above for $R_{YY}(\tau)$. The p.s.d. and a.c.f. of the output $Y(t)$ are shown in Fig. 11.2.

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Example 11.2 We now consider an example where the input is not w.s.s. Consider the LTI system given in the previous example with the impulse response stated there. Now suppose the same white noise process specified in the previous example is applied to the LTI system over the time period \([0,T]\) and the system is at rest, that is, has zero initial conditions, at \(t = 0\). Thus, the actual input to the system is given by:

\[
X(t) = W(t) [u(t) - u(t - T)] = \left\{ \begin{array}{ll}
W(t) & \text{for } t \in [0,T] \\
0 & \text{otherwise}
\end{array} \right. .
\tag{11.62}
\]

And, the output, as always with LTI systems, is the convolution of the input and the impulse response, given by

\[
Y(t) = X(t) \ast h(t)
\]

\[
= \int_{-\infty}^{\infty} X(t - \tau) h(\tau) \, d\tau
\tag{11.64}
\]

\[
= \int_{-\infty}^{\infty} X(\tau) h(t - \tau) \, d\tau .
\tag{11.65}
\]

We can proceed to determine the convolution in either one of the two forms.

Using the fact that \(X(t) = 0\) outside \([0,T]\), and is equal to \(W(t)\) over \([0,T]\), the first form of the convolution yields:

\[
Y(t) = \begin{cases}
0 & \text{for } t < 0 \\
\int_{0}^{t} h(\tau) X(t - \tau) \, d\tau & \text{for } 0 \leq t < T \\
\int_{-T}^{t} h(\tau) X(t - \tau) \, d\tau & \text{for } T \leq t
\end{cases}
\tag{11.66}
\]

\[
= \begin{cases}
0 & \text{for } t < 0 \\
\int_{0}^{t} e^{-\alpha \tau} W(t - \tau) \, d\tau & \text{for } 0 \leq t < T \\
\int_{-T}^{t} e^{-\alpha \tau} W(t - \tau) \, d\tau & \text{for } T \leq t
\end{cases} .
\tag{11.67}
\]

Similarly, the second form of the convolution integral yields the following expression for the output \(Y(t)\):

\[
Y(t) = \begin{cases}
0 & \text{for } t < 0 \\
\int_{0}^{t} X(\tau) h(t - \tau) \, d\tau & \text{for } 0 \leq t < T \\
\int_{0}^{T} X(\tau) h(t - \tau) \, d\tau & \text{for } T \leq t
\end{cases}
\tag{11.68}
\]
\[ R_{YY}(s, t) = E\left\{ \int_s^t W(\tau) e^{-\alpha(t-\tau)} d\tau \right\} \]  

(11.73)
\[ R_{YY}(s, t) = \begin{cases} 
 0 & \text{on A: } s < 0 \text{ or } t < 0 \\
 \frac{N_o}{2\alpha} e^{-\alpha t} (e^{\alpha s} - e^{-\alpha s}) & \text{on B: } 0 \leq s < T \text{ and } s \leq t \\
 \frac{N_o}{2\alpha} e^{-\alpha s} (e^{\alpha t} - e^{-\alpha t}) & \text{on C: } 0 \leq t < T \text{ and } t \leq s \\
 \frac{N_o}{2\alpha} e^{-\alpha(s+t)} (e^{2\alpha T} - 1) & \text{on D: } T \leq s \text{ and } T \leq t
\end{cases} \]
where the regions $A$, $B$, $C$, and $D$ are shown in Fig. 11.3. We clearly see from the functional expressions and the regions that $R_{Y|Y}(s,t)$ is not a function of $(s - t)$ only. Hence, the output process is not w.s.s., mainly because the input is not w.s.s.

EOE

11.5 DT Random Processes Transmitted through DT Systems

The definitions and formulation for the discrete-time (DT) case is essentially a special case of the general definitions and formulation. A DT signal (or random process) is one that is defined on integer-valued argument, such as, $x[n]$, or $X[n]$ for a DT random process. A DT system is one that accepts DT inputs and produces DT outputs: $x[n]$ input, $y[n]$ output, where $n$ is all integers. DT systems are also classified according to their properties. For example, a DT is said to be memoryless if

$$y[n_o] = g\{x[n_o], n_o\},$$

that is, $y[n_o]$ depends on $n_o$ and $x[n_o]$ only. A DT system is time-invariant if $x[n]$ producing $y[n]$ implies that $x[n - n_o]$ produces $y[n - n_o]$ for any $n_o$ and $x[n]$. A memoryless, time-invariant system is defined as one that satisfies

$$y[n_o] = g\{x[n_o]\}.$$  

When a DT random process $X[n]$ is applied to a memoryless, time-invariant DT system, the $N$ - th order CDF or PDF of the output random process $Y[n]$ can be found in terms of the CDF or PDF of the input process as a problem of $N$ functions of $N$ r.v.'s.
For a memoryless, time-invariant DT system, the stationarity results are analogous to the CT counterpart. Namely, if the input process \( X[n] \) is strictly stationary (or stationary of order \( k \)), then the output process \( Y[n] \) is also strictly stationary (or stationary of order \( k \)). And, if the input is w.s.s., then the output is not stationary in any sense.

### 11.5.1 DT Linear Systems

Similar to CT case, a DT linear system is one that satisfies the *superposition principle*: If the system at initial state \( A_1 \) with input \( x_1[n] \) produces output \( y_1[n] \) and at initial state \( A_2 \) with input \( x_2[n] \) produces output \( y_2[n] \), then at initial state \( c_1 A_1 + c_2 A_2 \) with input \( c_1 x_1[n] + c_2 x_2[n] \), it produces output \( c_1 y_1[n] + c_2 y_2[n] \), for any constants \( c_1 \) and \( c_2 \), any initial states \( A_1 \) and \( A_2 \), and inputs \( x_1[n] \) and \( x_2[n] \). Again, sometimes the reference to the initial conditions is left out of the definition of the superposition principle. The response (output) of a linear system can always be decomposed as the sum of the zero-state response and the zero-input response. Namely, \( y[n] = y_{zs}[n] + y_{zi}[n] \), where the zero-state response \( y_{zs}[n] \) is the response of the system to 0 initial conditions and input \( x[n] \), and the zero-input response \( y_{zi}[n] \) is the response of the system to initial conditions \( A \) and input 0.

DT linear systems are often described/characterized by linear difference equations. The solution of the difference equation corresponding to an initial condition and input (forcing function/sequence) corresponds to the output (response) of the system. Thus, techniques for solving difference equations in time domain or using \( z \)-transforms can be used to obtain the output of linear systems. We assume the reader is somewhat familiar with these techniques. For example, taking the \( z \)-transform of the difference equation and assuming zero initial conditions, yields an algebraic equation of the form:

\[
Y(z)Q(z) = X(z)P(z), \tag{11.93}
\]

where \( X(z) \) and \( Y(z) \) are \( z \)-transforms of \( x[n] \) and \( y[n] \), respectively, and \( Q(z) \) and \( P(z) \) are polynomials in \( z \). This algebraic equation involving the \( z \)-transforms can be rearranged to yield:

\[
H(z) \triangleq \frac{P(z)}{Q(z)} = \frac{Y(z)}{X(z)}, \tag{11.94}
\]

where \( H(z) \) is called the transfer function of the DT linear system. Thus, given or having obtained \( H(z) \) and given \( X(z) \), we can find \( Y(z) \) and taking its inverse \( z \)-transform, we can obtain \( y[n] \), the (zero-state) output of the system.
An alternative characterization of a DT linear system in the time-domain is obtained in terms of the so-called impulse response of the system. The impulse response of a DT linear system, denoted by $h[n, k]$, is defined as "the response of the system at time $n$ to an unit-sample (DT impulse) applied at $k$", with the assumption that the initial condition is zero. In other words, input $\delta[n - k]$ produces the output $h[n, k]$, with zero initial conditions. Now, we also know that, using the DT sifting property of the DT impulse ($\delta$ function), any input $x[n]$ can be represented as:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k].$$  \hspace{1cm} (11.95)

Now, we can invoke the superposition principle to say that if $\delta[n - k]$ produces $h[n, k]$, then $x[n]$ represented as the sum of $\delta[n - k]$'s with amplitude $x[k]$ produces

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n, k].$$  \hspace{1cm} (11.96)

Thus, we have a time domain expression for the output $y[n]$ in terms of the input $x[n]$ and the impulse response $h[n, k]$.

If the linear system is time-invariant, that corresponds to the impulse response $h[n, k]$ satisfying $h[n, k] = h[n - k]$. This means for a linear time-invariant (LTI) DT system, the response at $n$ due to an unit-sample at $k$, does not depend on $n$ and $k$ separately, but depends only on $(n - k)$, the difference of the two time variables, that is, the time (samples) elapsed since the $\delta$ was applied. When this is true, we can represent the impulse response $h[n - k]$ in terms of a single time variables $h[k]$, where $k$ now represents the time elapsed between the time the $\delta$ is applied and the time the output is observed. For a DT-LTI system, the above expression for the output becomes

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k],$$  \hspace{1cm} (11.97)

which we recognize as the DT convolution between the input $x[k]$ and the impulse response $h[k]$. Upon a simple change of variable, the convolution sum above can also be written as

$$y[n] = \sum_{k=-\infty}^{\infty} x[n - k] h[k].$$  \hspace{1cm} (11.98)

Thus, the DT convolution operation, denoted by $y[n] = x[n] \ast h[n]$, is symmetrical in the two components. We note again that the convolution between the input and the impulse response of a LTI system yields the (zero-state) response of the system.
A system is said to be causal or physically realizable if its output at time \( n \) depends on inputs at or prior to \( n \), not after \( n \). A system for which this is not true is called non-causal or physically unrealizable or anticipative, because it would be putting out a response due to an input before the input is applied. For a DT-LTI system, this property can be expressed in terms of the impulse response of the system: A LTI system is causal if and only if its impulse response satisfies:

\[
h[k] = 0, \quad \text{for } k < 0.
\]  

(11.99)

So, for a causal DT-LTI system the convolution sum for the output can be written as:

\[
y[n] = \sum_{k=-\infty}^{n} x[k] h[n-k]
\]

(11.100)

\[
y[n] = \sum_{k=0}^{\infty} x[n-k] h[k].
\]

(11.101)

Of course, if the input \( x[n] \) is zero on certain parts of the summation, then that would also change the limits of the summation. For example, if the input is zero until \( n = 0 \) and \( x[n] \) for \( n \geq 0 \), then the expression for the output would be

\[
y[n] = \sum_{k=0}^{n} x[k] h[n-k]
\]

(11.102)

\[
y[n] = \sum_{k=0}^{n} x[n-k] h[k],
\]

(11.103)

for \( n \geq 0 \); and, \( y[n] = 0 \), for \( n < 0 \).

Finally, we note that the \( z \)-transform of \( h[n] \), the impulse response of a LTI system, defined as

\[
H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n},
\]

(11.104)

is called the transfer function of the system, which was discussed in the context of the \( z \)-transform solution of a difference equation representing an LTI system. Depending on \( h[n] \), the \( z \)-transform \( H(z) \) exists only on certain parts of the \( z \) plane, which is called the region of convergence (ROC) of \( H(z) \). ROC is where the \( z \)-transform summation exists. For \( H(z) \) a rational function in \( z \) (or in \( z^{-1} \)), the associated ROC will be inside a circle, or outside a circle, or between two circles (an annular region) in the \( z \) plane. If the system is causal, then the ROC is outside of the circle passing through the poles of \( H(z) \) with largest magnitude. \( H(z) \) alone is not the \( z \)-transform of \( h[n] \); \( H(z) \) together with the associated ROC is the \( z \)-transform.
11.5. DT RANDOM PROCESSES TRANSMITTED THROUGH DT SYSTEMS

Because a single $H(z)$ with different ROC's correspond to different $h[n]$'s. Hence, the ROC associated with an $H(z)$ must always be specified to eliminate any ambiguity, unless the system is known to be causal, in which case the ROC is uniquely specified as described above. $h[n]$ and $\{H(z), \text{ROC}\}$ is a $z$-transform pair and satisfies certain properties.

The Discrete-Time Fourier Transform (DTFT) of $h[n]$ defined as

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}, \tag{11.105}$$

is called the system function or complex transfer function, when it exists. The system function $H(e^{j\omega})$ exists if $z = e^{j\omega}$, that is, the unit-circle in the $z$ plane, lies in the ROC associated with the $z$-transform. $h[n]$ and $H(e^{j\omega})$ (when it exists) is a DT-Fourier transform pair and satisfies all DTFT properties.

This completes a quick review of the essential properties of DT linear systems. For a more thorough treatment, the reader should refer to any Signals and Systems textbook used in Electrical Engineering curricula. In the next section we consider random processes going through linear systems.

11.5.2 Transmission of DT Random Processes through DT Linear Systems

We will now study the transmission of DT random processes through DT linear systems. We assume a certain characterization of the DT linear system perhaps in terms of its impulse response or the system function is given. Also given is a characterization of the input DT random process. Usually the characterization of the input random process is not in terms of individual sample functions, rather it is in terms of certain statistics of the input process, such as, its mean and autocorrelation function or power spectral density or joint PDF’s or CDF’s. And, the objective is to determine the corresponding statistics of the output random process.

Consider a DT linear time-invariant (LTI) system with impulse response $h[n]$. Suppose input to this system is a random process $X[n]$ with mean $\mu_X[n]$ and autocorrelation function (a.c.f.) $R_{XX}[m,n]$. (Note that at this point we do not assume $X[n]$ is w.s.s.; later we will.) Then the output of the system is a random process $Y[n]$ expressed as

$$Y[n] = \sum_{k=-\infty}^{\infty} X[k] h[n-k]. \tag{11.106}$$
We will now determine the mean and a.c.f. of the output process. First, its mean:

\[
\mu_Y[n] = \mathbb{E}\{Y[n]\} = \mathbb{E}\left\{ \sum_{k=-\infty}^{\infty} X[k] h[n-k] \right\} \tag{11.107}
\]

\[
= \sum_{k=-\infty}^{\infty} \mathbb{E}\{X[k]\} h[n-k] \tag{11.108}
\]

\[
= \sum_{k=-\infty}^{\infty} \mu_X[k] h[n-k] \tag{11.109}
\]

\[
= \mu_X[n] * h[n]. \tag{11.110}
\]

The second equality follows from the fact that expectation is a linear operation and since linear operations are commutative, the expected value can be moved inside the summation. Now, if the mean of the input process \(X[n]\) is constant, that is, \(\mu_X[n] = \mu_X\), which would be true if \(X[n]\) is w.s.s., then the above expression for \(\mu_Y[n]\) reduces to

\[
\mu_Y[n] = \mu_X \sum_{k=-\infty}^{\infty} h[n-k] \tag{11.111}
\]

\[
= \mu_X \sum_{k=-\infty}^{\infty} h[k] \tag{11.112}
\]

\[
= \mu_X H(e^{j\omega}), \tag{11.113}
\]

which is also a constant; that is, \(\mu_Y[n] = \mu_Y\). The second equality above follows from a change of variable in the summation and the last equality follows from letting \(\omega = 0\) in the DT Fourier transform of \(h[n]\).

We can find the a.c.f. of \(Y[n]\) in two steps (as we did in the CT case): First by finding the cross-correlation between \(X[n]\) and \(Y[n]\) and then finding the a.c.f. in terms of the cross-correlation:

\[
R_{YX}[m, n] = \mathbb{E}\{Y[m] X^*[n]\} \tag{11.114}
\]

\[
= \mathbb{E}\left\{ \sum_{k=-\infty}^{\infty} X[k] h[m-k] \cdot X^*[n] \right\} \tag{11.115}
\]

\[
= \sum_{k=-\infty}^{\infty} \mathbb{E}\{X[k] X^*[n]\} h[m-k] \tag{11.116}
\]

\[
= \sum_{k=-\infty}^{\infty} R_{XX}[k, n] h[m-k]. \tag{11.117}
\]

We recognize that the last expression is a convolution of \(h[m]\) and \(R_{XX}[m, n]\), with respect to
the first variable of $R_{XX}[m, n]$, that is,

$$R_{YX}[m, n] = R_{XX}[m, n] * h[m].$$  \hspace{1cm} (11.118)

In the second step, we obtain $R_{YY}[m, n]$ in terms of $R_{YX}[m, n]$ as follows:

$$R_{YY}[m, n] = E\{Y[m] Y^*[n]\} \hspace{1cm} (11.119)$$

$$= E\left\{ Y[m] \sum_{i=-\infty}^{\infty} X^*[i] h^*[n-i] \right\} \hspace{1cm} (11.120)$$

$$= \sum_{i=-\infty}^{\infty} E\{Y[m] X^*[i]\} h^*[n-i] \hspace{1cm} (11.121)$$

$$= \sum_{i=-\infty}^{\infty} R_{YX}[m, i] h^*[n-i] \hspace{1cm} (11.122)$$

$$= \sum_{i=-\infty}^{\infty} \left[ \sum_{k=-\infty}^{\infty} R_{XX}[k, i] h[m-k] \right] h^*[n-i]. \hspace{1cm} (11.123)$$

We recognize that the fourth equality is a convolution of $h^*[i]$ and $R_{YX}[m, i]$ with respect to the second variable of $R_{YY}[m, n]$, that is,

$$R_{YY}[m, n] = R_{YX}[m, n] * h^*[n] \hspace{1cm} (11.124)$$

$$= R_{XX}[m, n] * h[m] * h^*[n]. \hspace{1cm} (11.125)$$

Thus, we have obtained $R_{YY}[m, n]$, the a.c.f. of the output process $Y[n]$, in terms of the a.c.f. of the input process and the impulse response of the linear system.

Now, if we assume that the input process $X[n]$ is w.s.s., so that, $R_{XX}[k, n] = R_{XX}[k-n]$, then we obtain simplifications in the above expressions. The cross-correlation expression reduces to:

$$R_{YX}[m, n] = \sum_{k=-\infty}^{\infty} R_{XX}[k-n] h[m-k] \hspace{1cm} (11.126)$$

$$= \sum_{i=-\infty}^{\infty} R_{XX}[i] h[m-n-i]. \hspace{1cm} (11.127)$$

The second equality follows from a change of variable in the summation. We recognize that this expression does not depend on $m$ and $n$ separately, but depends on $(m-n)$, so we can express $R_{YX}[m, n]$ as $R_{YX}[m-n]$. We also recognize that the last expression is the convolution of $R_{XX}[k]$ with $h[k]$, that is,

$$R_{YX}[k] = R_{XX}[k] * h[k]. \hspace{1cm} (11.128)$$
Now substituting \( R_{YX}[m-k] \) in place of \( R_{YX}[m,k] \), the expression for \( R_{YY}[m,n] \), the a.c.f. of the output \( Y[n] \), in terms of \( R_{YX} \) becomes

\[
R_{YY}[m,n] = \sum_{k=\infty}^{\infty} R_{YX}[m-k] h^*[n-k] \\
= \sum_{i=\infty}^{\infty} R_{YX}[m-n-i] h^*[-i] .
\] (11.129)

(11.130)

We recognize that, the RHS of the second equality does not depend on \( m \) and \( n \) separately, but depends on \((m-n)\). Hence, we can express \( R_{YY}[m,n] \) as \( R_{YY}[m-n] \). We also recognize that the last expression is the convolution of \( R_{YX}[k] \) with \( h^*[-k] \). Recalling that \( R_{YX}[k] \) is the convolution of \( h[k] \) and \( R_{XX}[k] \), we finally have

\[
R_{YY}[k] = R_{XX}[k] \ast h[k] \ast h^*[-k] .
\] (11.131)

In obtaining the mean and a.c.f. of the output \( Y[n] \), we also obtained the result that with a LTI system if the input process is w.s.s. then the output process is also w.s.s. For the mean and a.c.f. we obtained:

\[
\mu_Y = H(e^{j\omega}) \cdot \mu_X \\
R_{YY}[k] = R_{XX}[k] \ast h[k] \ast h^*[-k] .
\] (11.132)

(11.133)

Using the convolution property of DT Fourier transform pairs on the expression for \( R_{YY}[k] \), we obtain the relationship between the input and output power spectral densities:

\[
S_{YY}(e^{j\omega}) = \left| H(e^{j\omega}) \right|^2 S_{XX}(e^{j\omega}) \\
= |H(e^{j\omega})|^2 S_{XX}(e^{j\omega}) .
\] (11.134)

(11.135)

Other stationarity results that hold for DT linear time-invariant systems are as follows:

1. If the input \( X[n] \) to a DT-LTI system is stationary in the strict sense, then the output is also stationary in the strict sense.

2. If the input \( X[n] \) to a DT-LTI system is stationary of order \( N \), then in general the output is not stationary in any sense (or order).

We stress the fact that the output is w.s.s. and the spectral analysis to obtain the p.s.d. of the output are true, only if the input is w.s.s. If the input to the LTI system is not w.s.s. then
the spectral analysis does not hold. For example, consider a LTI system and a w.s.s. process $X[n]$. If $X[n]$ is applied to the system starting at a finite time or until a finite time or over a finite time interval, then the actual input to the system is NOT w.s.s., although $X[n]$ is w.s.s. This is because the input is not $X[n]$ over all time, it is $X[n]$ only over part of the time, hence it is not w.s.s. This would be the case when the system is represented by a linear difference equation with a w.s.s. $X[n]$ which is applied to the system at some initial time (which is not $-\infty$) with zero or non-zero initial conditions. The input to the system in this case is zero upto initial time and $X[n]$ after that, hence the input is NOT w.s.s. In all such cases where the input is not w.s.s., we have to obtain an expression for the output which may consist of different expressions valid at different time intervals and proceed to find its mean and a.c.f. of the output using this multi-line expression. The mean may or may not be constant and the a.c.f. $R_{yy}[m, n]$ will not be a function of $(m - n)$ only. Thus, the output will not be w.s.s. in general. Below we will do two examples: first, where the input is w.s.s. and we can do the spectral analysis, and, second, where the input is not w.s.s. and therefore the output is not w.s.s.

**Example 11.3** Consider a DT-LTI system with impulse response

$$h[n] = a^n u[n] , \quad |a| < 1. \quad (11.136)$$

The system (complex transfer) function for this system is

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}. \quad (11.137)$$

Suppose a zero mean, DT white noise process $W[n]$ with a.c.f.

$$R_{ww}[k] = N_0 \delta[k] \quad (11.138)$$

is applied as input. The power spectral density (p.s.d.) of the input is

$$S_{ww}(e^{j\omega}) = N_0, \quad \text{for all } \omega . \quad (11.139)$$

Now we want to find the mean, a.c.f., and p.s.d. of the output $Y[n]$. We know that the system is LTI and the input is w.s.s., hence the output will also be w.s.s. and we can use the spectral analysis. First, the mean:

$$E\{Y[n]\} = E\{W[n]\} H(e^{j0}) = 0 , \quad (11.140)$$
since $E\{W[n]\} = 0$. Using the relation between the input and output p.s.d.'s and assuming that $a$ is real, we obtain

\[
S_{YY}(e^{j\omega}) = \frac{|H(e^{j\omega})|^2 S_{WW}(e^{j\omega})}{N_o (1 - 2a \cos \omega + a^2)}.
\]

The inverse DTFT of this p.s.d. can be obtained as

\[
R_{YY}[k] = \frac{N_o}{1 - a^2} a^{|k|}, \quad \text{for all } k.
\]

Thus, we find the a.c.f. of the output using its p.s.d.

Note that we can also find the a.c.f. $R_{YY}[k]$ through the double convolution:

\[
R_{YY}[k] = R_{WW}[k] \ast h[k] \ast h^*[-k]
\]

The convolution with the $\delta[k]$ is trivial and the convolution of the two geometric sequences (one right-sided, one left-sided) yields the two-sided geometric sequence given above for $R_{YY}[k]$. The p.s.d. and a.c.f. of the output $Y[n]$ are shown in Fig. 11.4.
Example 11.4 We now consider an example where the input is not w.s.s. Consider the DT-LTI system given in the previous example with the impulse response stated there. Now suppose the same DT white noise process specified in the previous example is applied to the DT-LTI system starting at $n = 0$, and the system is at rest, that is, $Y[n] = 0$, for $n < 0$. Thus, the actual input to the system is given by:

$$X[n] = W[n] u[n] = \begin{cases} W[n] & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases} \quad (11.146)$$

And, the output is the convolution of the input and the impulse response, given by

$$Y[n] = X[n] \ast h[n]$$

$$= \sum_{k=-\infty}^{\infty} X[n-k] h[k]$$

$$= \sum_{k=-\infty}^{\infty} X[k] h[n-k] \quad (11.149)$$

We can proceed to use either one of the two forms of convolution sum.

Since $X[n] = 0$ and $h[n] = 0$ for $n < 0$, and $X[n] = W[n]$ for $n \geq 0$, the second expression for the convolution sum can be written as:

$$Y[n] = \begin{cases} 0 & \text{for } n < 0 \\ \sum_{k=0}^{n} W[k] a^{n-k} & \text{for } n \geq 0 \end{cases}$$

$$= \begin{cases} 0 & \text{for } n < 0 \\ a^{n} \sum_{k=0}^{n} W[k] a^{-k} & \text{for } n \geq 0 \end{cases} \quad (11.151)$$

We now determine the mean and a.c.f. of $Y[n]$. First, the mean:

$$E\{Y[n]\} = \begin{cases} E\{0\} & \text{for } n < 0 \\ a^{n} \sum_{k=0}^{n} E\{W[k]\} a^{-k} & \text{for } n \geq 0 \end{cases}$$

$$\equiv 0 \quad , \quad (11.153)$$

since $E\{W[k]\} \equiv 0$ for all $k$. But if $E\{W[k]\} = \mu \neq 0$, then clearly we would get a time-varying $E\{Y[n]\}$, which would imply that $Y[n]$ is not w.s.s.

Now, let’s find the a.c.f. $R_{YY}[m,n]$:

$$R_{YY}[m,n] = E\{Y[m]Y^*[n]\} \quad . \quad (11.154)$$
It follows from the above expression for $Y[n]$ that $R_{YY}[m,n] = 0$ if $m < 0$ or $n < 0$ or both. For $m \geq 0$ and $n \geq 0$, it is given by

$$R_{YY}[m,n] = E \left\{ \left( \sum_{i=0}^{m} W[i] a^{m-i} \right) \cdot \left( \sum_{k=0}^{n} W[k] a^{n-k} \right) \right\}$$

(11.155)

$$= \sum_{i=0}^{m} \sum_{k=0}^{n} a^{(m-i+n-k)} E \{W[i] W^*[k]\}$$

(11.156)

$$= \sum_{i=0}^{m} \sum_{k=0}^{n} a^{(m-i+n-k)} N_o \delta[i-k]$$

(11.157)

$$= N_o a^{(m+n)} \sum_{k=0}^{\min(m,n)} a^{-2k}$$

(11.158)

$$= N_o a^{(m+n)} \frac{1 - (a^{-2})^{\min(m,n)+1}}{1 - a^{-2}}.$$  

(11.159)

Now combining all cases, we have the a.c.f. of the output $Y[n]$ as

$$R_{YY}[m,n] = \begin{cases} 
0 & \text{for } m < 0 \text{ or } n < 0 \\
N_o a^{n-m} \left( \frac{a^{m+2}-1}{a^2-1} \right) & \text{for } n \geq m \geq 0 \\
N_o a^{m-n} \left( \frac{a^{n+2}-1}{a^2-1} \right) & \text{for } m \geq n \geq 0
\end{cases}$$

(11.160)

This a.c.f. $R_{YY}[m,n]$ clearly is not a function of $(m-n)$ only. Hence, the output $Y[n]$ is not a w.s.s. process. This is mainly due to the fact that the actual input $X[n]$ is not w.s.s.

11.6 Problems

1. A wide-sense stationary zero-mean random process $X(t)$ has the autocorrelation function

$$R_{XX}(\tau) = \frac{\sin(\tau)}{\tau}.$$  

(The power spectral density $S_{XX}(j\Omega)$ of $X(t)$ was determined in Prob. 10.20.) If $X(t)$ is input to an ideal lowpass filter with system (complex transfer) function

$$H(j\Omega) = \begin{cases} 
A & \text{for } |\Omega| < \frac{1}{4} \\
0 & \text{otherwise}
\end{cases}$$

where $A$ is a constant. Find the autocorrelation function and the power spectral density of the output $Y(t)$.

2. A w.s.s. process $X(t)$ with mean $\mu$ and autocorrelation $R_{XX}(\tau) = \mu^2 + e^{-\alpha|\tau|}$ is transmitted through an ideal lowpass filter with frequency response

$$H(j\Omega) = \begin{cases} 
A & \text{for } |\Omega| \leq B \\
0 & \text{otherwise}
\end{cases}.$$  

Find
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a) An expression for the output $Y(t)$ in terms of $X(t)$.
b) The mean of $Y(t)$.
c) The power spectral density of $Y(t)$.
d) The autocorrelation function of $Y(t)$. (You may leave this in integral form.)

3. Suppose $X(t)$, a w.s.s. random process, is input to a LTI system; the output $Y(t)$ is given by:

$$Y(t) = \int_{t-T}^{t} X(\tau) d\tau.$$ 

a) Find $S_{YY}(j\Omega)$ in terms of $S_{XX}(j\Omega)$. (Hint: First identify $h(\tau)$, the impulse response; then find $H(j\Omega)$, the frequency response, of the system.)
b) Find $S_{YY}(j\Omega)$ and $E\{|Y(t)|^2\}$, if $R_{XX}(\tau)$ is given as

$$R_{XX}(\tau) = \begin{cases} 1 - \frac{|\tau|}{T} & \text{for } |\tau| \leq T \\ 0 & \text{otherwise} \end{cases}$$

4. Consider a LTI system with impulse response $h(\tau)$. The system is causal, i.e., $h(\tau) = 0$ for $\tau < 0$. Also consider a w.s.s. random process $V(t)$ with mean $\mu$ and a.c.f. $R_{VV}(\tau)$. The input to the system is

$$X(t) = \begin{cases} V(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}.$$ 

Determine the following quantities, in terms of $V(t)$ and its given statistics:

a) The output $Y(t)$, for all $t$.
b) The mean of $Y(t)$.
c) The a.c.f. of $Y(t)$.

5. Suppose $W(t)$ is a zero mean white noise process with autocorrelation $R_{WW}(\tau) = N_o \delta(\tau)$. Also consider a LTI system with impulse response $h(\tau) = B[u(\tau) - u(\tau - T)]$. The system is at rest until $t = 0$. Starting at $t = 0$, $W(t)$ is applied as the input. $Y(t)$ is the output of the system.

a) Determine $Y(t)$ in terms of $W(t)$, $B$, and $T$, for all $t$.
b) Determine the cross-correlation $R_{WY}(s,t)$, for all $s$ and $t$.
c) Determine the variance of $Y(t)$ for all $t$.

6. Consider an LTI system with impulse response $h(\tau) = u(\tau) - u(\tau - T)$.

a) Find the a.c.f. of the output of the system, when the input is $W(t)$, a zero-mean white noise process with a.c.f. $N_o \delta(\tau)$, applied starting at $-\infty$.
b) Find the a.c.f. of the output if the system is at rest at $t = 0$ and the input is $W(t)[u(t) - u(t - T)]$, where $W(t)$ is the white noise specified in part (a).

7. Suppose $W(t)$, a zero-mean white noise process with autocorrelation function $R_{WW}(\tau) = N_o \delta(\tau)$, is applied to a LTI system over the interval $0 \leq t \leq 3$. The system impulse response is $h(\tau) = e^{-2\tau} u(\tau)$; the system is at rest at $t = 0$. Let $Y(t)$ be the output for all $t$. Write down an expression for $Y(t)$ for all $t$. Find $R_{YY}(s,t)$, the autocorrelation function of the output, for all $(s,t)$.
8. Suppose \(c + W(t)\), where \(c\) is a constant and \(W(t)\) is a white noise process with mean zero and autocorrelation function \(R_{WW}(t) = \alpha \delta(t)\) is applied as input to a LTI system with impulse response \(h(t) = e^{-t} u(t)\), starting at \(t = 0\). Upto \(t = 0\), the input is 0. Let \(Y(t)\) represent the output for all \(t\).

   a) Write down an expression for \(Y(t)\) for all \(t\).
   b) Find \(E\{Y(t)\}\), the mean of \(Y(t)\), for all \(t\).
   c) Find \(R_{YY}(s, t)\), the autocorrelation function of \(Y(t)\), for all \(s\) and \(t\).
   d) Is \(Y(t)\) w.s.s.?

9. Random process \(X(t)\), defined below, is input to a linear time-invariant system

   \[
   X(t) = g(t) + W(t)
   \]

   where \(g(t) = u(t) - u(t - T)\), \(u(t)\) is unit-step function, and \(W(t)\) is a zero-mean white Gaussian noise with autocorrelation function \(R_{WW}(\tau) = \alpha \delta(\tau)\). The impulse response of the system is given as

   \[
   h(t) = e^{-t} u(t)
   \]

   and \(Y(t)\) is the output of the system.

   a) Determine the mean and autocorrelation of \(X(t)\). Is \(X(t)\) w.s.s.?
   b) Write down an explicit expression for \(Y(t)\) for all \(t\).
   c) Find the mean of \(Y(t)\).
   d) Find the autocorrelation of \(Y(t)\).
   e) Is \(Y(t)\) w.s.s.?

10. A linear system is described by the differential equation

   \[
   Y''(t) + 2Y'(t) = X(t)
   \]

   where \(X(t)\) is the input and \(Y(t)\) is the output. First find the transfer function and the impulse response of the LTI system represented by the DE. Then find \(R_{XY}(t_1, t_2)\) and \(R_{YY}(t_1, t_2)\), when

   a) \(X(t)\) is a zero-mean white noise process with autocorrelation function

   \[
   R_{XX}(\tau) = N_0 \delta(\tau)
   \]

   applied to the system starting at \(t = -\infty\).
   b) \(X(t)\) is zero until \(t = 0\) and is the white noise given in part (a) for \(t \geq 0\) and the system is at rest at \(t = 0\).

11. Consider the system shown in Fig. 11.5 with input random process \(X(t)\). Suppose \(X(t)\) is zero mean white noise process with autocorrelation function \(R_{XX}(\tau) = N_0 \delta(\tau)\) and the impulse responses for the two systems are given as

   \[
   h_1(t) = u(t) - u(t - 3),
   h_2(t) = e^{-2t} u(t).
   \]

   a) Find \(R_{YY}(\tau)\) and \(S_{YY}(j\Omega)\).
   b) Find \(R_{YZ}(s, t)\), the cross correlation, and \(S_{YZ}(j\Omega)\), the cross power spectral density of \(Y(t)\) and \(Z(t)\), if \(Y(t)\) and \(Z(t)\) happen to be jointly stationary.
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Figure 11.5: The system described in Example 11.11.

c) If the white noise is applied to \( H_2 \) over \( t \in [0, 3] \) find the autocorrelation function of the output of \( H_2 \), assuming that the system was at rest at \( t = 0 \).

12. A DT process \( X[n] \) consisting of iid r.v.'s with mean \( \mu \) and variance \( \sigma^2 \) is input to a DT-LTI system with impulse response \( h[n] = u[n] - u[n - 3] \).

a) Find the autocorrelation function and the power spectral density of the input \( X[n] \).

b) Find the mean, autocorrelation function and power spectral density of \( Y[n] \), the output of the DT system.

13. Suppose \( \{X_n\} \) is a DT-RP where \( \{X_n\} \) are iid and uniformly distributed on \([-1,1]\) and \( \{X_n\} \) is applied (for all \( n \)) as input to a LTI discrete-time system with impulse response \( h[n] = \delta[n] - 0.5\delta[n - 1] + 0.2\delta[n - 2] \).

a) Determine the mean, autocorrelation and power spectral density of the input \( \{X_n\} \).

b) Determine the mean, autocorrelation and power spectral density of \( \{Y_n\} \), the output of the system.

14. Consider a DT-LTI system with impulse response

\[
h[n] = \begin{cases} \frac{1}{N} & \text{for } n = 0, 1, \ldots, N - 1 \\ 0 & \text{otherwise} \end{cases}
\]

a) Write down an expression for the output of the system, if the input is \( W[n] \), a zero-mean white noise process with \( E\{W^2[n]\} = N_0 \), applied to the system starting at \( n = -\infty \). Find the a.c.f. of the output.

b) Write down an expression for the output of the system, if the system is at rest at \( n = 0 \) and the input is

\[
X[n] = \begin{cases} W[n] & \text{for } n = 0, 1, \ldots, N - 1 \\ 0 & \text{otherwise} \end{cases}
\]

where \( W[n] \) is the white noise sequence specified in part (a).

c) Find the a.c.f. of the output.
15. Let \( \{X_n\} \) be a discrete w.s.s. process with mean zero and autocorrelation function 
\[ R_{XX}[m] = \alpha^{|m|} \] 
where \( |\alpha| < 1 \), and let \( h[n] = u[n] - u[n-2] \) be the impulse response of a LSI-DT system. The system is at rest (i.e., at zero-state) up to \( n = 0 \) and \( \{X_n\} \) is applied to the system as input starting at \( n = 0 \); let \( \{Y_n\} \) represent the output of the system.

a) Determine an expression for \( \{Y_n\} \) for all \( n \).
b) Determine the cross-correlation \( R_{XY}[m, n] \) for all \( m \) and \( n \).
c) Determine \( R_{YY}[m, n] \) the autocorrelation function of the output for all \( m \) and \( n \).

16. Suppose \( X_c(t) \) is a zero-mean w.s.s. Gaussian process with autocorrelation function 
\[ R_c(\tau) = \frac{A}{\pi \tau} \sin(2\pi 10^3 \tau) \] 
and power spectral density 
\[ S_c(j\Omega) = \begin{cases} 
A & \text{for } |\Omega| \leq 2\pi 10^3 \text{ rads./sec.} \\
0 & \text{otherwise} 
\end{cases} \]

\( X_c(t) \) is sampled with an impulse train of period \( T = 2.5 \times 10^{-4} \) sec. to yield the discrete-time random process \( X[n] = X_c(nT) \). (In Prob. 10.15, the autocorrelation function and the power spectral density of the discrete-time process \( X[n] \) was determined.)

If \( X[n] \) is input to a linear time-invariant discrete-time system with impulse response 
\( h[n] = \delta[n] - 0.5 \delta[n-1] \) and \( Y[n] \) is the output, find the power spectral density and the autocorrelation function of the output. Determine \( R_{YY}[0] \) and \( R_{YY}[3] \).

17. Consider the DT-LTI system with impulse response 
\( h[k] = a^k u[k] \), where \( |\alpha| < 1 \). Also consider the DT process \( X[n] = \mu + W[n] \), where \( \mu \) is a constant and \( W[n] \) is a zero-mean discrete-time white noise sequence with \( E\{W^2[n]\} = N_0 \). Suppose the system is at rest at \( n = 0 \) and \( X[n] u[n] \) is applied as the input.

a) Determine the output \( Y[n] \).
b) Determine the expected value of the output.
c) Determine the a.c.f of the output.

18. Suppose \( \{X_n\} \), a discrete white noise process, with mean zero and variance \( \sigma_X^2 \), is input to a LSI-DT system described by the difference equation (DE):
\[ Y_n = \alpha Y_{n-1} + X_n, \quad |\alpha| < 1. \]

a) Write down expressions for \( R_X[k] \) and \( S_X(e^{j\omega}) \).
b) Taking the z-transform of the above DE, one obtains \( Y(z) = \alpha z^{-1} Y(z) + X(z) \). Using this, obtain \( H(z) = \frac{Y(z)}{X(z)} \), the transfer function; and, by inverse z-transforming \( H(z) \), obtain \( h[n] \), the impulse response of the system.
c) Using results of parts a) and b), determine the autocorrelation function \( R_Y[k] \) and the power spectral density \( S_Y(e^{j\omega}) \) of the output process \( \{Y_n\} \).

19. Consider the system shown in Fig. 11.6, with the random process \( X[n] \) as input. Suppose \( X[n] \) is a zero mean discrete-time white noise process with autocorrelation function 
\[ R_{XX}(n) = N_0 \delta[n] \] 
and the impulse responses for the two systems are given as
\[ h_1[n] = u[n] - u[n-3], \]
\[ h_2[n] = a^n u[n], \quad |\alpha| < 1. \]
a) Find $R_{YY}[n]$ and $S_{YY}(e^{j\omega})$.

b) Find $R_{YZ}[m,n]$, the cross correlation, and $S_{YZ}(e^{j\omega})$, the cross power spectral density of $Y[n]$ and $Z[n]$, if $Y[n]$ and $Z[n]$ happen to be jointly stationary.

c) If the process $X[n](u[n] - u[n - 3])$ is applied to $H_2$, find the autocorrelation function of the output of $H_2$, assuming that the system was at rest for $n < 0$. 

Figure 11.6: The system described in Example 11.19.