

Elasticity equations in polar coordinates (See section 3.7):

Transformation equations:

$$\begin{aligned} x &= r \cos \theta & r^2 &= x^2 + y^2 \\ y &= r \sin \theta & \theta &= \tan^{-1}(y/x) \end{aligned}$$

Derivatives and differentials:

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{r} = \cos \theta & \frac{\partial r}{\partial y} &= \frac{y}{r} = \sin \theta \\ \frac{\partial \theta}{\partial x} &= -\frac{y}{r^2} = -\frac{\sin \theta}{r} & \frac{\partial \theta}{\partial y} &= \frac{x}{r^2} = \frac{\cos \theta}{r} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial}{\partial \theta} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \end{aligned}$$

Equilibrium:

$$\begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + F_r &= 0 \\ \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + F_\theta &= 0 \end{aligned}$$

Strain-displacement:

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r}, \quad \gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{v}{r}$$

Hooke's law:

Plane stress

$$\epsilon_r = (\sigma_r - \nu \sigma_\theta)/E, \quad \epsilon_\theta = (\sigma_\theta - \nu \sigma_r)/E, \quad \gamma_{r\theta} = \tau_{r\theta}/G$$

Plane strain

$$\begin{aligned} \epsilon_r &= (1 + \nu)[(1 - \nu)\sigma_r - \nu\sigma_\theta]/E \\ \epsilon_\theta &= (1 + \nu)[(1 - \nu)\sigma_\theta - \nu\sigma_r]/E \\ \gamma_{r\theta} &= \tau_{r\theta}/G \end{aligned}$$

Compatibility (zero body forces):

$$\frac{\partial^2 \epsilon_\theta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \epsilon_r}{\partial \theta^2} + \frac{2}{r} \frac{\partial \epsilon_\theta}{\partial r} - \frac{1}{r} \frac{\partial \epsilon_r}{\partial \theta} = \frac{1}{r} \frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial \gamma_{r\theta}}{\partial \theta}$$

Stress function:

$$\sigma_r = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}, \quad \sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right)$$

$$\nabla^4 \Phi = \nabla^2 \nabla^2 \Phi = 0$$

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)$$