

Orthotropic elasticity in 3D:

In three dimensions the stress-strain relations are expressed as $\sigma = c\epsilon$, which, expanded into components, is

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix}. \quad (1)$$

To move from stresses to strains, the relationship can be written as $\epsilon = d\sigma$ with $d = c^{-1}$ the inverse of the constitutive matrix c . In isotropic elasticity, there are three elastic constants, E , ν , and G , the elastic modulus, Poisson's ratio, and shear modulus of which two are independent. The components of d are given by

$$d_{11} = d_{22} = d_{33} = \frac{1}{E} \quad (2)$$

$$d_{12} = d_{13} = d_{23} = d_{21} = d_{31} = d_{32} = -\frac{\nu}{E} \quad (3)$$

$$d_{44} = d_{55} = d_{66} = \frac{1}{G}. \quad (4)$$

Orthotropic materials have 3 planes of material symmetry, and 9 independent elastic constants. These correspond to elastic moduli in each of the three principal material directions, three Poisson's ratios that represent the coupling between axial and transverse strains, and three shear moduli that represent the relationship between the three modes of shear stress and strain. These elastic constants are arranged in the d matrix as follows

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{xy}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{yz}} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}. \quad (5)$$

These twelve elastic constants are subject to constraints because the constitutive matrix must be symmetric if the strain energy density is to exist and properly relate the stresses and strains. The constraints are $\nu_{xy}E_y = \nu_{yx}E_x$, $\nu_{xz}E_z = \nu_{zx}E_x$ and $\nu_{yz}E_z = \nu_{zy}E_y$. Note that when the coordinate system in which the stress and strain are represented coincides with the coordinate system in which the constitutive matrix is represented, the shear and axial components of stress and strain are decoupled. When these coordinate systems do not coincide, however, shear and axial components do not remain uncoupled, and axial stress can lead to shear strain.

Example Exercise: Consider an orthotropic material with elastic constitutive matrix of Eq. 5 whose constitutive matrix is expressed in the coordinate system (x, y, z) . A coordinate system (x', y', z') is oriented such that the angle from x to x' is $\pi/4$, and the z and z' axes are coincident. The state of stress in the (x', y', z') coordinate system is $\sigma_{x'} = \sigma_0$ with all other stress components zero. Find the strains in the (x, y, z) and (x', y', z') coordinate systems, express an effective elastic modulus $E_{x'} = \sigma_{x'}/\epsilon_{x'}$ and comment on your results.

Solution: This can be treated as a 2D problem. The transformed stresses, using $\theta = -\pi/4$ are $\sigma_x = \sigma_0/2$, $\sigma_y = \sigma_0/2$ and $\tau_{xy} = \sigma_0/2$ with all other components equal to zero. The strains in the (x, y, z) coordinate system are $\epsilon_x = \frac{\sigma_0}{2E_x} - \frac{\nu_{yx}\sigma_0}{2E_y}$, $\epsilon_y = \frac{\sigma_0}{2E_y} - \frac{\nu_{xy}\sigma_0}{2E_x}$ and $\gamma_{xy} = \frac{\sigma_0}{2G_{xy}}$.

To get the strains in the (x', y', z') coordinate system, we can use the 2D transformation equations with $\theta = \pi/4$. These are $\epsilon_{x'} = \frac{\sigma_0}{4} \left(\frac{1}{E_x} - \frac{2\nu_{yx}}{E_y} + \frac{1}{E_y} + \frac{1}{G_{xy}} \right)$, $\epsilon_{y'} = \frac{\sigma_0}{4} \left(\frac{1}{E_x} - \frac{2\nu_{yx}}{E_y} + \frac{1}{E_y} - \frac{1}{G_{xy}} \right)$ and $\gamma_{x'y'} = \frac{\sigma_0}{2} \left(\frac{1}{E_y} - \frac{1}{E_x} \right)$. Note here that if $E_x = E_y$ the shearing strain becomes zero and the axial and shear behavior is decoupled.

These strains lead to the effective modulus $E_{x'} = \frac{\sigma_{x'}}{\epsilon_{x'}} = \frac{4E_x E_y G_{xy}}{E_y G_{xy} - 2\nu_{yx} E_x G_{xy} + E_x G_{xy} + E_x E_y}$. This reduces to $E_{x'} = E$ if $E_x = E_y = E$, $\nu_{yx} = \nu$, and $G_{xy} = E/(2(1 + \nu))$, that is, we recover the isotropic case with the appropriate material property substitutions.