

University of Massachusetts - Amherst
 Department of Civil & Environmental Engineering
 CEE 630: Advanced Solid Mechanics (Fall 2006)

Midterm Exam (1.5 hours): October 26

Problem 1: (15 pts) Consider the state of stress

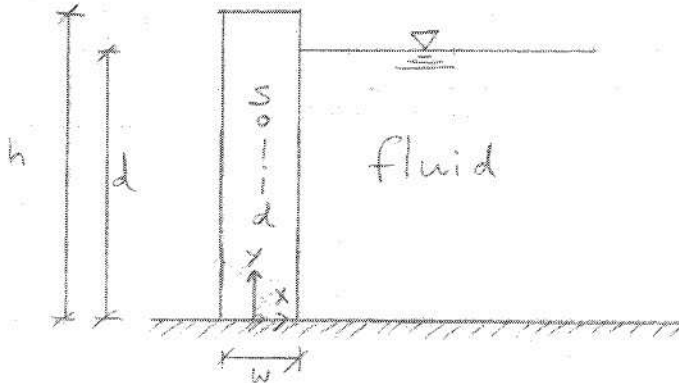
$$\sigma = \begin{bmatrix} 6 & -3 \\ -3 & -2 \end{bmatrix}$$

which is defined in an orthogonal coordinate system (x, y) and given in units of ksi.

(a) Draw a representative material element with these stresses shown. Draw the coordinate system.

(b) Suppose that this state of stress exists in a structure that has a failure stress in tension and compression of 10ksi and in shear of 8 ksi. Is this structure safe?

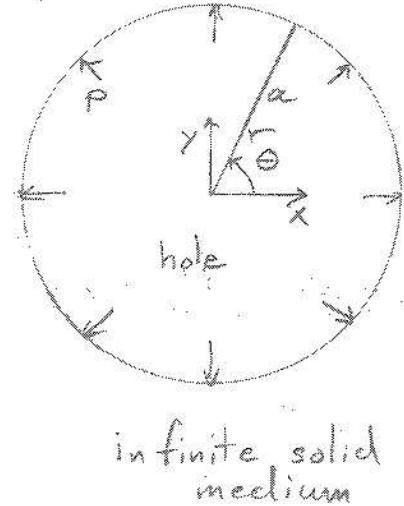
Problem 2: (15 pts) The unit thickness block shown below is in a state of static equilibrium. Write down the complete set of traction boundary conditions you would use in solving for the state of stress in the block. Be clear about any assumptions you make



about the traction distributions.

Problem 3: State the assumptions made in formulating plane stress problems. Explain why plane stress solutions are only approximate.

Problem 4: (15 pts) The state of stress in the body shown below is $\sigma_x = 0, \sigma_y =$



$\sigma_0, \tau_{xy} = 0$. Under the assumptions of plane stress, find the displacements.

Problem 5: (10 pts) Two of the classes of equations we have studied are equilibrium and compatibility equations. For each of these classes, give an example, describe what kind of equations they are, and explain their physical meaning.

Problem 6: (15 pts) The state of stress in a material is given by

$$\begin{aligned}\sigma_x &= c_1x + c_2y + c_3x^2 \\ \sigma_y &= c_4x^2 + c_5y^2 \\ \tau_{xy} &= c_6y + c_7xy\end{aligned}$$

- Check equilibrium of these stresses and determine relationships between the constants c_1 to c_7 .
- Rewrite the stresses by substituting for the constants using the relations derived in (a).
- Determine a stress function that corresponds to these stresses. Take careful note of constants and functions of integration.

Problem 7: (15 pts) The figure below shows three cross sections. One a solid bar, one a hollow tube, and one a flat plate. The values of J for these three cross sections is are given by $J = \pi r^4/2$, $J = 2\pi r_0^3 t$, and $J = bt^3/3$. Note that the second of these is approximate, and applies only when $r_0 \gg t$. Assume $t = 1$ for the tube and plate.

- Calculate r_0 for the tube, and b for the plate so that all three cross sections have the same value of J . Your answers should be in terms of r , the radius of the solid bar.
- Calculate the cross sectional area of each section. For the tube, you may approximate the area by the circumference times the thickness.
- Investigate and comment upon the relative area of the three cross sections as $r \rightarrow \infty$. Can you conclude anything about the relative efficiency of the three cross sections in resisting torsion?

Useful expressions in polar coordinates:
Transformation equations:

$$\begin{aligned}x &= r \cos \theta & r^2 &= x^2 + y^2 \\y &= r \sin \theta & \theta &= \tan^{-1}(y/x)\end{aligned}$$

Derivatives and differentials:

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{x}{r} = \cos \theta & \frac{\partial r}{\partial y} &= \frac{y}{r} = \sin \theta \\ \frac{\partial \theta}{\partial x} &= -\frac{y}{r^2} = -\frac{\sin \theta}{r} & \frac{\partial \theta}{\partial y} &= \frac{x}{r^2} = \frac{\cos \theta}{r}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial}{\partial \theta} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}\end{aligned}$$

Equilibrium:

$$\begin{aligned}\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + F_r &= 0 \\ \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + F_\theta &= 0\end{aligned}$$

Strain-displacement:

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{u}{r}, \quad \gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r} \frac{\partial u}{\partial \theta} - \frac{v}{r}$$

Hooke's law:

Plane stress

$$\epsilon_r = (\sigma_r - \nu \sigma_\theta)/E, \quad \epsilon_\theta = (\sigma_\theta - \nu \sigma_r)/E, \quad \gamma_{r\theta} = \tau_{r\theta}/G$$

Plane strain

$$\begin{aligned}\epsilon_r &= (1 + \nu)[(1 - \nu)\sigma_r - \nu\sigma_\theta]/E \\ \epsilon_\theta &= (1 + \nu)[(1 - \nu)\sigma_\theta - \nu\sigma_r]/E \\ \gamma_{r\theta} &= \tau_{r\theta}/G\end{aligned}$$

Compatibility (zero body forces):

$$\frac{\partial^2 \epsilon_\theta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \epsilon_r}{\partial \theta^2} + \frac{2}{r} \frac{\partial \epsilon_\theta}{\partial r} - \frac{1}{r} \frac{\partial \epsilon_r}{\partial r} = \frac{1}{r} \frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial \gamma_{r\theta}}{\partial \theta}$$

Stress function:

$$\sigma_r = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}, \quad \sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right)$$

$$\nabla^4 \Phi = \nabla^2 \nabla^2 \Phi = 0$$

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right)$$