Solutions to ECE 673 Test 2

(1a) Let \( r(t) \) be the residual time to the next bus arrival, \( A(t) \) the number of arrivals over \([0, t]\), and \( X_i \) the interarrival time between the \( i-1 \)st and ith buses. The average residual time is

\[
\bar{R} = \lim_{t \to \infty} \frac{1}{t} \int_0^\infty r(\tau) d\tau = \lim_{t \to \infty} \frac{A(t) \int_0^\infty r(\tau) d\tau}{A(t)} = \lim_{t \to \infty} \frac{A(t) \sum_{i=1}^{\infty} X_i^2}{2A(t)} = \frac{E[X^2]}{2E[X]}
\]

\[
E[X^2] = 0.1 \int_{10}^{20} x^2 \, dx = 700/3
\]

\[
E[X] = 15
\]

\[
\bar{R} = \frac{700}{90} = 7.78
\]

(1b) Let \( N \) be the number of buses that are full. Then, the expected waiting time is

\[
E[W] = \bar{R} + 15E[N].
\]

To find \( N \), we have to find the probability, \( p_f \), that the first bus is full. Since we arrive at a random point in time, this probability is the steady-state probability that a bus is full. This can be obtained by considering the following Markov chain:

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  Full  Not Full

  0.8   0.1  0.2
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We have

\[
0.8p_f = 0.1p_e
\]

\[
p_f = 1 - p_e
\]

where \( p_f, p_e \) are the probability of the bus being full and empty, respectively. Solving these equations yields \( p_f = 1/9; p_e = 8/9 \). Hence, the probability that we will have to
miss N buses is given by:

\[ p_m(N) = \begin{cases} 
  p_c & \text{if } N = 0 \\
  (0.2)^{N-1} 0.8p_f & \text{otherwise}
\end{cases} \]

We then have

\[ E[N] = 0.8p_f \sum_{N=1}^{\infty} N (0.2)^{N-1} = 0.8p_f \frac{1}{(0.8)^2} = 0.138. \]

The expected waiting time is therefore given by

\[ 7.78 + 0.138 \times 15 = 9.85. \]

(2) \( B^*(s) = e^{-10s}. \) Denote the mean busy period by \( g. \) Hence, we have

\[ G^*(s) = e^{-10(s + \lambda G^*(s))} \]

\[ \Rightarrow \frac{dG^*(s)}{ds} = -10 \left( 1 - \lambda \frac{dG^*(s)}{ds} \right) e^{-10(s + \lambda G^*(s))} \]

\[ \Rightarrow \left. \frac{dG^*(s)}{ds} \right|_{s=0} = -10 \left( 1 - \lambda \frac{dG^*(s)}{ds} \right) e^{-10(s + \lambda G^*(s))} \bigg|_{s=0} \]

\[ \Rightarrow g = -10(1 + \lambda g) \]

\[ \Rightarrow g = \frac{10}{1 - 10\lambda} \]

(3) There are only two possible states here: \((0, 1), (1, 0),\) where \((i, j)\) means there are \(i\) in the first system and \(j\) in the second. The balance equation is \( \mu_1 p_{1,0} = \mu_2 p_{0,1}. \) Combining this with the fact that the probabilities must add to one, we get the utilizations, \( u_1, u_2: \)

\[ u_1 = p_{1,0} = \frac{\mu_2}{\mu_1 + \mu_2} \]

\[ u_2 = p_{0,1} = \frac{\mu_1}{\mu_1 + \mu_2} \]

(4) The service time is the sum of two exponentials. We can consider the service to have two stages, \( A \) and \( B, \) in which the time spent is exponentially distributed with parameters \( \mu_1 \) and \( \mu_2, \) respectively. The Markov chain is shown below and the balance equations can be written by inspection of it.