ECE 673: Solutions to Homework 5

(2) The probability that output line \( j \) is busy is

\[
\pi_j = 1 - \prod_{i=0}^{n-1} (1 - p_{ij}).
\]

The expected bandwidth is the sum of the probabilities of the lines being busy:

\[
E[BW] = \sum_{j=0}^{m-1} \pi_j.
\]

(3) Similar to an M/M/1 Markov chain, except that the rate from state \( i \) to state \( i - 1 \) is \( i \mu \). The differential equations are:

\[
\frac{d\pi_0(t)}{dt} = -\lambda \pi_0(t) + \mu \pi_1(t)
\]

\[
\frac{d\pi_i(t)}{dt} = -(\lambda + i \mu) \pi_i(t) + \lambda \pi_{i-1}(t) + (i + 1) \mu \pi_{i+1}(t), \quad i > 0
\]

(4) A busy period begins upon an arrival to an empty queue and lasts until the next visit of that queue to state 0. So, we can simply alter the M/M/∞ Markov chain by removing all transitions out of state 0, i.e., by making state 0 an absorbing state. Write the differential equations for such a chain. Solve the differential equations with the initial condition that \( \pi_1(0) = 1 \). Then, the probability distribution function of the busy period is given by \( G(t) = \pi_0(t) \). (Can you see why this works?)

(5a) The crosswalk is being modeled as an M/M/∞ queue. One way of solving for the probability of coming upon an empty queue is to write out the balance equations for the queue and then solving for \( \pi_0 \). We can solve the equations either directly or by using z-transforms.

Solving the equations directly yields \( p_i = (\rho^i / i!) p_0 \) where \( \rho = \lambda / \mu \). Using the condition that the probabilities must sum to one, we end up with \( p_0 = e^{-\rho} \).

Using z-transforms is a bit more complicated. As usual, multiply the i'th balance equation by \( z^i \) on both sides and then add the equations. We will get:

\[
\lambda p_0 = \mu p_1
\]

\[
(\lambda + \mu) p_1 z = \lambda p_0 z + 2 \mu p_2 z
\]

\[
(\lambda + 2 \mu) p_2 z^2 = \lambda p_1 z^2 + 3 \mu p_3 z^2
\]

\[\vdots\]

\[
(\lambda + i \mu) p_i z^i = \lambda p_{i-1} z^{i-1} + (i + 1) \mu p_{i+1} z^{i}
\]
After some minor manipulation, we obtain the following differential equation:

\[
\lambda (1 - z) P(z) = \mu (1 - z) \frac{dP(z)}{dz}
\]

\[\Rightarrow \frac{dP(z)}{P(z)} = \rho \, dz\]

\[\Rightarrow \ln P(z) = \rho z + K, \text{ } K \text{ is a constant of integration}\]

\[\Rightarrow P(z) = e^{\rho z + K}\]

\[P(1) = 1 \Rightarrow K = -\rho \Rightarrow P(z) = e^{-\rho (1-z)}\]

\[\Rightarrow p_0 = P(0) = e^{-\rho}\]

There is a much simpler way to calculate \( p_0 \). The average time between the end of one busy period and the beginning of another is \( t_{idle} = 1/\lambda \). The average duration of a busy period is given to be \( \theta \). So, we have alternating busy and idle periods, with mean times \( t_{idle} \) and \( \theta \), respectively. Hence the probability that we will arrive at a random instant and find the crosswalk empty of pedestrians is

\[p_0 = \frac{t_{idle}}{t_{idle} + \theta}\]

(A byproduct of this calculation is that since we also know that \( p_0 = e^{-\rho} \), we now have an expression for the average busy period, \( \theta \), in this queue.)

The probability that we do not need to stop at all is given by \( p_0^5 \).

(5b) We have to calculate the average residual time of the busy period. We already know how to calculate average residual times: see the discussion of the vacation model. The average residual time is given by \( \phi/(2\theta) \) (in the vacation model, we showed it was \( V^2/(2\theta) \)). This is the delay if we have to stop; if not, the waiting time is \( \theta \). Hence, the total expected time is

\[5(1 - p_0) \frac{\phi}{2\theta}\].