

Homework #2

Due 3/15/06

1. Consider the normed linear space consisting of the real line \mathcal{R} with norm

$$\|x\| = \begin{cases} 0, & x = 0 \\ 1, & \text{else} \end{cases} .$$

(a) Show that this is a Banach space.

(b) Consider any normed linear space X with an arbitrary norm and let $f : \mathcal{R} \rightarrow X$ be a mapping from the real line with norm defined above to X . Show that the function f is continuous.

2. Consider the normed linear space consisting of the set X of all real sequences $x = \{x_1, x_2, \dots\}$ such that:

$$\sum_{i=1}^{\infty} i x_i^2 < \infty,$$

with norm defined as:

$$\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2},$$

Show that this does **not** form a Banach space. *Hint: Consider sequences of the form $\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, 0, \dots\}$. Also, you might find the following useful:*

$$\sum_{k=1}^{\infty} k^{-a} = \begin{cases} \infty, & a \leq 1 \\ \text{finite}, & a > 1 \end{cases} .$$

3. Consider the normed linear space of real-valued continuous functions on $[0, 1]$ with $\|x\| = \max\{|x(t)| : 0 \leq t \leq 1\}$. Show that there is **not** an inner product on the space that induces this norm. One possible (and probably the easiest) method is to show that the parallelogram equality is not satisfied.

4. Consider the set $\{x_1, x_2, \dots, x_n\}$ of linearly independent vectors. Prove that the Gram-Schmidt process generates an orthonormal basis for the space spanned by $\{x_1, x_2, \dots, x_n\}$.
5. Consider a set of ideal low-pass filters whose frequency responses are given by:

$$H_n(f) = \begin{cases} 1, & |f| \leq nf_0 \\ 0, & \text{else} \end{cases}$$

where f_0 is some fixed frequency. Let $h_n(t)$ be the impulse response of the n^{th} filter, and consider the subspace of $L_2(-\infty, \infty)$ spanned by $\{h_1(t), h_2(t), \dots, h_n(t)\}$. Find an orthonormal basis for this subspace. (*Hint: Recall the general version of Parseval's Theorem given by $\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$).*

6. (a) Find an orthonormal basis for the subspace of $L_2[0, 1]$ spanned by the functions $\{1, t, t^2, t^3\}$.

(b) Let $y(t) = \sin(\pi t)$, $0 \leq t \leq 1$, and let $\hat{y}(t)$ denote the best 3rd-order polynomial approximation to $y(t)$ in the sense that $\|\hat{y}(t) - y(t)\|$ is minimized. Find and sketch $\hat{y}(t)$.